

# Symmetries of Analytic Curves

Maximilian Hanusch\*

*Department of Physics  
Florida Atlantic University  
777 Glades Road  
FL 33431 Boca Raton  
USA*

January 25, 2016

## Abstract

Analytic curves are classified w.r.t. their symmetries under a regular Lie group action on an analytic manifold. We show that an analytic curve is either exponential or splits into countably many analytic immersive curves; each of them decomposing naturally into symmetry free subcurves mutually and uniquely related by the group action. We conclude that a connected analytic 1-dimensional submanifold is either analytically diffeomorphic to the unit circle or some interval, or that each point (except for at most countably many) admits a symmetry free chart.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Preliminaries</b>	<b>5</b>
2.1	Conventions . . . . .	5
2.2	Analytic curves . . . . .	6
2.2.1	Basic properties . . . . .	6
2.2.2	Relations between curves . . . . .	8
2.2.3	Self relations of curves . . . . .	9
2.3	Reparametrizations . . . . .	10
2.4	Regularity and stabilizers . . . . .	12
2.4.1	Regularity . . . . .	12
2.4.2	Stabilizers . . . . .	14
2.5	Lie algebra generated curves . . . . .	15
2.5.1	Standard facts . . . . .	15
2.5.2	Uniqueness . . . . .	16
<b>3</b>	<b>The Classification</b>	<b>18</b>
3.1	Lie curves . . . . .	18
3.2	Free curves . . . . .	19
3.3	The regular case . . . . .	22

---

\*e-mail: [hanuschm@fau.edu](mailto:hanuschm@fau.edu)

<b>4</b>	<b>Decompositions</b>	<b>29</b>
4.1	Basic properties . . . . .	29
4.2	Existence . . . . .	33
4.3	Non-compact decompositions . . . . .	36
4.4	The compact case . . . . .	38
4.5	Arbitrary domains . . . . .	43
<b>5</b>	<b>Extension: Analytic 1-Manifolds</b>	<b>45</b>

## 1 Introduction

The basic configuration observables of loop quantum gravity [1, 2] are holonomies along embedded analytic curves. In [4] symmetries of such curves have been studied for the purpose of investigating quantum reduced configuration spaces occurring there; and in the present paper, these results are generalized to regular Lie group actions and arbitrary analytic curves. Here, we will follow the lines of [4], whose investigations have been based on the concept of a free segment. More concretely, given a Lie group action  $\varphi: G \times M \rightarrow M$  on an analytic manifold that is analytic in  $G$  and  $M$ , an analytic immersive curve<sup>1</sup>  $\gamma: D \rightarrow M$  is said to be a free segment iff

$$g \cdot \gamma \sim_{\circ} \gamma \quad \text{for } g \in G \quad \implies \quad g \cdot \gamma = \gamma.$$

Here,  $g \cdot \gamma \sim_{\circ} \gamma$  means that  $g \cdot \gamma(J) = \gamma(J')$  holds for non-empty open intervals  $J, J'$  on which  $\gamma$  is an embedding. Then, for  $\mathfrak{g}_{\gamma}$  the Lie algebra of the stabilizer  $G_{\gamma} := \bigcap_{t \in D} G_{\gamma(t)}$  of  $\gamma$ , our classification result states that, cf. Theorem 3.6

### Theorem

*If  $\varphi$  is regular, an analytic curve  $\gamma$  is either free or Lie. Thus, in the latter case, of the form*

$$\gamma: t \mapsto \exp(\rho(t) \cdot \vec{g}) \cdot x \quad \forall t \in \text{dom}[\gamma] \quad (1)$$

*for some  $x \in M$ , some  $\vec{g} \in \mathfrak{g}$ , and some analytic map  $\rho: \text{dom}[\gamma] \rightarrow D \subseteq \mathbb{R}$ . Here, if  $\gamma$  is non-constant analytic and Lie w.r.t.  $x \in M$  and  $\vec{g} \in \mathfrak{g}$ , it is Lie w.r.t. some further  $y \in M$  and  $\vec{q} \in \mathfrak{g}$  iff  $y \in \exp(\text{span}_{\mathbb{R}}(\vec{g})) \cdot x$  and  $\vec{q} \in \lambda \cdot \vec{g} + \mathfrak{g}_{\gamma}$  holds for some  $\lambda \neq 0$ .<sup>2</sup>*

Here, free means that  $\gamma|_D$  is a free segment for some interval  $D \subseteq \text{dom}[\gamma]$ , and  $\varphi$  is called regular iff the following two conditions hold, cf. Definition 2.4.1

- i) If  $x \notin C \subseteq M$  with  $|C| \geq 2$ , then there exists a neighbourhood  $U$  of  $x$  with  $g \cdot C \not\subseteq U$  for each  $g \in G$ .
- ii) If  $\lim_n g_n \cdot x = x$  holds for  $\{g_n\}_{n \in \mathbb{N}} \subseteq G \setminus G_x$ , then  $\{h_n \cdot g_n \cdot h'_n\}_{n \in \mathbb{N}}$  has a convergent subsequence for some  $\{h_n\}_{n \in \mathbb{N}}, \{h'_n\}_{n \in \mathbb{N}} \subseteq \bigcap_{g \in G} G_{g \cdot x}$ ; the stabilizer of the orbit  $G \cdot x$ .

For instance, pointwise proper, hence proper actions are regular. Moreover,  $\varphi$  is regular if the following two conditions hold, cf. Remark 2.15

- $M$  is a topological group with  $\varphi(g, x) = \phi(g) \cdot x$  for some continuous group homomorphism  $\phi: G \rightarrow M$ .
- $\phi \circ s = \text{id}_V$  holds for a continuous map  $s: V := U \cap \phi(G) \rightarrow G$ , for  $U$  some neighbourhood of  $e_M$ .

In particular, the above theorem applies to the case where  $G$  (or a closed subgroup) acts in the natural way on  $M = G/H$ , for  $H$  some closed normal subgroup of  $G$ . Indeed, the next two examples even show that regularity is more general than pointwise properness, because the respective subgroups  $H$  are not compact there:

<sup>1</sup>More precisely,  $D$  is an interval with non-empty interior, and  $\gamma$  is the restriction to  $D$  of an analytic immersion  $\tilde{\gamma}: I \rightarrow M$  defined the open interval  $I$ .

<sup>2</sup>The more general statements concerning non-constant Lie algebra generated curves  $\gamma_{\vec{g}}^x: t \mapsto \exp(t \cdot \vec{g}) \cdot x$ , can be found in Subsection 2.5.2. For instance, we have  $\gamma_{\vec{q}}^x = \gamma_{\vec{g}}^x$  iff  $\vec{q} \in \vec{g} + \mathfrak{g}_{\gamma}$  holds for  $\gamma := \gamma_{\vec{g}}^x$ .

- $G = \mathbb{R}^n$  and  $H = \mathbb{Z}^n$ , hence  $M = \mathbb{T}^n$ ,
- $G = \mathbb{R}^n$  and  $H \subseteq \mathbb{R}^n$  some  $m$ -dimensional linear subspace for  $m > 0$ , hence  $M = \mathbb{R}^{n-m}$ .

The above Theorem 3.6, without the second uniqueness statement, has originally been proven in [4] for embedded analytic curves with compact domain, namely for the situation where  $\varphi$  is analytic, admits only normal stabilizers, and is proper or transitive and pointwise proper, cf. Proposition 5.23 in [4]. Then, in [3], extensive technical efforts have been made to generalize this statement to the analytic pointwise proper case (no uniqueness statement).

The more general Theorem 3.6 now follows by elementary arguments from Lemma 3.3, stating that an analytic immersive curve is locally of the form (1) if it fulfils some local approximation property. Indeed, we then will first derive from i) that each non-free curve has a special self similarity property. This is done in Lemma 3.12 and Corollary 3.13, which basically reflect the argumentations in Lemma 5.19.2 in [4]. Then, we will conclude from i) and ii) that this self similarity property implies the mentioned approximation property, which is the content of Subsection 3.3. The second uniqueness statement in Theorem 3.6, is proven in Corollary 2.27.

Now, given a connected 1-dimensional analytic submanifold  $(S, \iota)$  of  $M$  with boundary,<sup>3</sup> each chart  $(U, \psi)$  of  $S$  with  $U$  connected, and  $\iota(U)$  contained in the domain of a chart of  $M$ , defines the analytic immersive curve  $\gamma_\psi := \iota \circ \psi^{-1}$ . Then, we easily obtain (cf. Proposition 5.1) that, if one such  $\gamma_\psi$  is Lie w.r.t. to some  $\vec{g} \in \mathfrak{g} \setminus \mathfrak{g}_x$  for  $x \in M$ , each such curve is Lie w.r.t.  $\vec{g}$  and  $x$ ; and then  $(S, \iota)$  is either analytically diffeomorphic to  $U(1)$  or to some interval  $D \subseteq \mathbb{R}$  via<sup>4</sup>

$$e^{i\phi} \mapsto \iota^{-1}(\exp(\phi \cdot \vec{g}) \cdot x) \quad \text{or} \quad t \mapsto \iota^{-1}(\exp(t \cdot \vec{g}) \cdot x),$$

respectively, whereby  $\vec{g}$  has to be suitably scaled in the first case. In particular, defining  $(S, \iota)$  to be free/Lie iff some  $\gamma_\psi$  is free/Lie, we easily conclude that, cf. Corollary 5.2

### Corollary

*If  $\varphi$  is regular,  $(S, \iota)$  is either free or Lie, with what each  $\gamma_\psi$  is free or Lie, respectively.*

Now, in addition to the classification Theorem 3.6, we will show that each free analytic immersive curve  $\gamma$  decomposes naturally into free segments, mutually and uniquely related by the group action. This has been proven in Proposition 5.23 in [4] for embedded analytic curves with compact domain, and worked out in little more detail<sup>5</sup> in [3]. More precisely, we will show, cf. Theorem 4.23 (for  $\text{dom}[\gamma]$  an arbitrary interval, cf. Theorem 4.28)

### Theorem

*Let  $\gamma: I \rightarrow M$  be an analytic immersion which is free but not a free segment, and assume that  $\varphi$  fulfils i). Then,  $\gamma$  either admits a unique  $\tau$ -decomposition or a compact maximal interval. In the second case,  $\gamma$  is either positive or negative, and admits a unique  $A$ -decomposition for each compact maximal  $A$ .*

Here,  $D \subseteq I$  is called maximal iff it is maximal w.r.t. the property that  $\gamma|_D$  is a free segment. Then, each such interval is necessarily closed in  $I = (i', i)$ , hence either compact or of the form  $(i', \tau]$  or  $[\tau, i)$  for some  $\tau \in I$ . Moreover, given analytic immersions  $\gamma: D \rightarrow M$  and  $\gamma': D' \rightarrow M$  with  $\gamma|_{\text{dom}[\mu]} = \gamma' \circ \mu$  for some analytic diffeomorphism  $\mu$ , we will write  $\gamma \rightsquigarrow \gamma'$  iff one of the following situations holds:

- ▷  $D$  and  $D'$  are compact, and  $\mu: D \rightarrow D'$ .
- ▷  $D$  is compact,  $\text{dom}[\mu] \subset D$  as well as  $D' = \text{im}[\mu]$  are half-open, and  $\text{dom}[\mu] \cap \partial D$  is singleton.
- ▷  $D = (i', \tau]$  and  $D' = [\tau, i)$  are half-open, and  $\mu(\tau) = \tau$  as well as  $\text{dom}[\mu] = D$  or  $\text{im}[\mu] = D'$  holds.

Then, the above theorem can be understood in the following way:

<sup>3</sup>This means a connected analytic 1-manifold  $S$  with boundary, together with an injective analytic immersion  $\iota: S \rightarrow M$ .

<sup>4</sup>Respective uniqueness statements concerning the Lie algebra element  $\vec{g}$  also hold in this case.

<sup>5</sup>It was figured out that at most two group elements are necessary to relate the different free segments, and formulas have been provided for the two cases discussed there, see also a) and b) below.

- I) If there is no compact maximal  $A \subseteq I$ , then  $(i', \tau]$ ,  $[\tau, i)$  are the only maximal intervals for some necessarily unique  $\tau \in I$ . Moreover, there is a unique class<sup>6</sup>  $[g] \neq [e]$ , such that  $g \cdot \gamma|_{(i', \tau]} \rightsquigarrow \gamma|_{[\tau, i)}$  holds w.r.t.  $\mu$ , whereby a proper translate of  $\gamma|_{(i', \tau]}$  can overlap  $\gamma$  only in this way.

More precisely, if  $g' \cdot \gamma|_J = \gamma \circ \rho$  holds for some analytic diffeomorphism  $\rho: (i', \tau] \supseteq J \rightarrow J'$ , then we either have

$$[g'] = [e] \quad \text{and} \quad \bar{\rho}|_{(i', \tau]} = \text{id}_{(i', \tau]} \quad \text{or} \quad [g'] = [g] \quad \text{and} \quad \bar{\rho}|_{\text{dom}[\mu]} = \mu$$

for  $\bar{\rho}$  the maximal analytic immersive extension of  $\rho$ .

For instance, if  $\text{SO}(2)$  acts via rotations on  $\mathbb{R}^2$ , then  $\gamma: \mathbb{R} \ni t \mapsto (t, t^3)$  admits the 0-decomposition  $[g_\pi]$ , for  $g_\pi$  the rotation by the angle  $\pi$ , cf. Example 4.15.

- II) If there is a compact maximal interval  $A$ , each translate of  $\gamma|_A$  overlaps  $\gamma$  in a unique way.

More precisely, there exists  $(\{a_n\}_{n \in \mathbf{n}}, \{[g_n]\}_{n \in \mathbf{n}})$  unique ( $A$ -decomposition) with  $\{a_n\}_{n \in \mathbf{n}} \subseteq I$ ,  $A = [a_{-1}, a_1]$ , and  $[g_{\pm 1}] \neq [e]$ , such that

$$g_n \cdot \gamma|_A \rightsquigarrow \gamma|_{A_n} \quad \text{holds w.r.t. an analytic diffeomorphism } \mu_n \text{ for all } n \in \mathbf{n}.$$

Here,  $\mathbf{n} = \{n \in \mathbb{Z}_{\neq 0} \mid \mathbf{n}_- \leq n \leq \mathbf{n}_+\}$  holds for some  $-\infty \leq \mathbf{n}_- < 0 < \mathbf{n}_+ \leq \infty$  with  $I = \bigcup_{n \in \mathbf{n}} A_n$  for the intervals

$$\begin{aligned} A_{\mathbf{n}_-} &:= (i', a_{\mathbf{n}_-}] \quad \text{if } \mathbf{n}_- \neq -\infty & A_{\mathbf{n}_+} &:= [a_{\mathbf{n}_+}, i) \quad \text{if } \mathbf{n}_+ \neq \infty \\ A_n &:= [a_{n-1}, a_n] \quad \text{for } \mathbf{n}_- < n \leq -1 & A_0 &:= [a_{-1}, a_1] & A_n &:= [a_n, a_{n+1}] \quad \text{for } 1 \leq n < \mathbf{n}_+. \end{aligned}$$

Then, for  $\mu_0 := \text{id}_A$ ,  $g_0 := e$ , and  $\rho$  some analytic diffeomorphism with  $\text{dom}[\rho] \subseteq A$ , we have

$$g \cdot \gamma|_{\text{dom}[\rho]} = \gamma \circ \rho \quad \implies \quad [g] = [g_n] \quad \text{and} \quad \bar{\rho}|_{\text{dom}[\mu_n]} = \mu_n \quad \text{for } n \in \mathbf{n} \sqcup \{0\} \text{ unique.} \quad (2)$$

Moreover, in the situation of II), two different cases can occur

- a) In the first case ( $\gamma$  positive, cf. Proposition 4.19), for each compact maximal  $A$ , the respective diffeomorphisms  $\mu_n$  are positive ( $\dot{\mu}_n > 0$ ), and each compact  $A_n$  is maximal. Moreover, there is some unique class  $[h]$ , such that  $[g_n] = [h^n]$  holds for all  $n \in \mathbf{n}$ , and this class is the same for each compact maximal  $A$ . In addition to that, each  $t \in I$  is contained in the interior of such an  $A$ .

For instance, if  $\mathbb{R}$  acts via  $\varphi(t, (x, y)) := (t + x, y)$  on  $\mathbb{R}^2$ , then  $\gamma: \mathbb{R} \ni t \mapsto (t, \sin(t))$  is positive, with compact maximal intervals  $[t, t + 2\pi]$  for each  $t \in \mathbb{R}$ , and  $[h] = [2\pi]$ , cf. Example 4.20.

- b) In the second case, ( $\gamma$  negative, cf. Proposition 4.22), the derivative of the diffeomorphism  $\mu_n$  has the signature  $(-1)^n$ , and  $[g_n] = [g_{\sigma(\text{sign}(n))} \cdot \dots \cdot g_{\sigma(n)}]$  holds for all  $n \in \mathbf{n}$ . Here,  $\sigma: \mathbb{Z}_{\neq 0} \rightarrow \{-1, 1\}$  is defined by

$$\sigma(n) := \begin{cases} (-1)^{n-1} & \text{if } n > 0 \\ (-1)^n & \text{if } n < 0, \end{cases}$$

so that  $[g_{\pm 2}] = [g_{\pm 1} \cdot g_{\mp 1}]$ ,  $[g_{\pm 3}] = [g_{\pm 1} \cdot g_{\mp 1} \cdot g_{\pm 1}]$  holds, and so on. Finally, for  $A$  some compact maximal interval, each of the intervals  $A_n$  is maximal, and they are the only maximal ones. Thus, if  $B$  is any other negative interval, it equals some  $A_n$ , and then the respective  $B$ -decomposition can be obtained from the  $A$ -decomposition by using Property (2).

For instance, if the euclidean group  $\mathbb{R}^2 \rtimes \text{SO}(2)$  acts on  $\mathbb{R}^2$  in the canonical way, then  $\gamma: \mathbb{R} \ni t \mapsto (t, \sin(t))$  is negative with compact maximal interval  $A = [0, \pi]$ . In this case,  $[g_{-1}]$  and  $[g_1]$  are classes of the rotations by  $\pi$  around  $(0, 0)$  and  $(\pi, 0)$ , respectively, cf. Example 4.21.  $\ddagger$

Then, for  $(S, \iota)$  as above, we conclude that, cf. Corollary 5.4

<sup>6</sup>We define  $[g] := g \cdot G_\gamma$  for  $G_\gamma := \{h \in G \mid h \cdot \gamma = \gamma\}$  the stabilizer of  $\gamma$ .

### Corollary

If  $\varphi$  is regular and  $(S, \iota)$  is free, then except for at most countably many points, each  $z \in S$  admits a neighbourhood  $V \subseteq S$ , such that  $g \cdot \iota(V) \cap \iota(V)$  is finite for each  $g \in G \setminus G_S$ .

Here,  $G_S := \bigcap_{s \in S} G_s$  denotes the stabilizer of  $S$ , and the countably many exception points are basically<sup>7</sup> given by the splitting points  $\{a_n\}_{n \in \mathbb{N}}$  which correspond to  $A$ -decompositions of each negative  $\gamma_\psi$ .

Finally, for some non constant analytic  $\gamma$ , the set  $Z = \{t \in \text{dom}[\gamma] \mid \dot{\gamma}(t) = 0\}$  consists of isolated points, and admits no limit point in  $\text{dom}[\gamma]$ , just by analyticity of  $\dot{\gamma}$ . Thus,  $\gamma$  splits canonically into countably many analytic immersive subcurves, “pinned together” at the points in  $Z$ , cf. Remark 3.8. Then, each of these subcurves is free as well, cf. Corollary 3.7, so that our decomposition results apply to each of them separately. Anyhow, besides certain combinatorial and technical issues, a deeper investigation of the analysis of  $\gamma$  at the points in  $Z$  seems to be necessary to prove analogous decomposition results also for the general non constant analytic case. For connected analytic 1-submanifolds, the strategy is sketched in the end of Section 5, and the to expected results are stated there. [5]

This paper is organized as follows:

- In Section 2, we fix the notations and collect the basic facts and definitions we will need in the main text.
- In Section 3, we prove our classification Theorem 3.6.
- In Section 4, we prove our decomposition results for analytic immersive curves, cf. Theorem 4.23 and 4.28.
- In Section 5, connected analytic 1-submanifolds are discussed. We prove the Corollaries 5.2 and 5.4, and pave the way for global decomposition results for such manifolds.

## 2 Preliminaries

In this section, we will fix some conventions and provide several basic facts and definitions that we will need to work efficiently in the main text. Let us start with some

### 2.1 Conventions

Manifolds will always be assumed to be second countable Hausdorff and analytic. If  $f: M \rightarrow N$  is a differentiable map between the manifolds  $M$  and  $N$ , by  $df: TM \rightarrow TN$ , we will denote the respective differential map between their tangent manifolds. The differentiable map  $f$  is said to be immersive iff for each  $x \in M$ , the restriction  $d_x f := df|_{T_x M}: T_x M \rightarrow T_{f(x)} N$  is injective. Elements of tangent spaces will usually be written with arrows, such as  $\vec{v} \in T_x M$ .

By an interval, we will understand a connected subset  $D \subseteq \mathbb{R}$  with non-empty interior  $\text{int}[D]$ . We will say that  $-\infty$  or  $\infty$  is a boundary point of  $D$  iff  $\inf[D] = -\infty$  or  $\sup[D] = \infty$  holds, respectively. If we write  $I, J$  or  $K, L$  instead of  $D$ , we will always mean that  $I, J$  are open, and that  $K, L$  is compact.

A curve is a continuous map  $\gamma: D \rightarrow X$  between an interval  $D$  and a topological space  $X$ . Then,

- If  $t \in \text{int}[D]$  holds, then  $\gamma|_{D \cap (-\infty, t]}$  and  $\gamma|_{D \cap [t, \infty)}$  are called initial and final segments (of  $\gamma$ ), respectively.
- If  $\gamma$  is injective, then  $\gamma^{-1}: \text{im}[\gamma] \rightarrow \text{dom}[\gamma]$  will denote its inverse in the sense of mappings.

An extension of  $\gamma$ , is a curve  $\tilde{\gamma}: I \rightarrow X$  defined on an open interval  $I$  containing  $D$ , such that  $\tilde{\gamma}|_D = \gamma$  holds. If  $M$  is an analytic manifold, the curve  $\gamma: D \rightarrow M$  is said to be

- analytic iff it admits an analytic extension.
- (analytic) immersive iff it admits an (analytic) immersive extension.
- an analytic embedding iff it admits an analytic immersive extension which a homeomorphism onto its image equipped with the relative topology.

<sup>7</sup>If  $z$  is a boundary point of  $S$ , we always find a neighbourhood as in the above corollary, cf. Remark 5.5.

Similarly, a function  $\tilde{\rho}: I \rightarrow I'$  is said to be an extension of the function  $\rho: D \rightarrow D'$  iff  $\rho = \tilde{\rho}|_D$  holds. Then,  $\rho$  is said to be

- analytic iff it admits an analytic extension.
- an (analytic) diffeomorphism iff it admits an extension which is an (analytic) diffeomorphism.

A diffeomorphism  $\rho: D \rightarrow D'$  is said to be positive or negative iff  $\dot{\rho}(t) > 0$  or  $\dot{\rho}(t) < 0$  holds for one, and the each  $t \in \text{int}[D]$ , respectively.

Now,  $\varphi: G \times M \rightarrow M$  will always denote a left action of a Lie group  $G$  on manifold  $M$ . Here, we will always assume that  $\varphi$  is analytic in  $G$  and  $M$ , i.e., that the maps

$$\varphi_x: G \rightarrow G, \quad g \mapsto \varphi(g, x) \quad \text{and} \quad \varphi_g: M \rightarrow M, \quad x \mapsto \varphi(g, x)$$

are analytic for each  $x \in M$ , and each  $g \in G$ . We will write  $g \cdot x$  instead of  $\varphi(g, x)$  if it helps to simplify the notations. Then, if  $x \in M$  is fixed,

- $G_x = \{g \in G \mid g \cdot x = x\}$  will denote its stabilizer, and  $\mathfrak{g}_x$  the Lie algebra of  $G_x$ .
- $G \cdot x = \{g \cdot x \mid g \in G\}$  will denote the orbit of  $x$  under  $G$ , having the stabilizer  $G_{[x]} := \bigcap_{g \in G} G_{g \cdot x}$ .

Finally, for  $x \in M$  and  $\vec{g} \in \mathfrak{g}$ , we define the analytic curve

$$\gamma_{\vec{g}}^x: \mathbb{R} \rightarrow M, \quad t \mapsto \exp(t \cdot \vec{g}) \cdot x, \quad (3)$$

which is analytic immersive iff  $\vec{g} \notin \mathfrak{g}_x$  holds, and constant elsewhere, cf. Lemma 2.21. We will say that an analytic curve  $\gamma: D \rightarrow M$  is **Lie** iff  $\gamma = \gamma_{\vec{g}}^x \circ \rho$  holds for some  $x \in M$ , some  $\vec{g} \in \mathfrak{g}$ , and some analytic map  $\rho: D \rightarrow D' \subseteq \mathbb{R}$ . Then,

- $\gamma|_{D'}$  is analytic immersive for some interval  $D' \subseteq D$  iff  $\rho|_{D'}$  is a diffeomorphism, and  $\vec{g} \notin \mathfrak{g}_x$  holds.
- each constant analytic curve is Lie.

## 2.2 Analytic curves

This subsection collects the most important properties of analytic curves that we will need.

### 2.2.1 Basic properties

Let us start with the straightforward observation that

#### Lemma 2.1

If  $\gamma, \gamma': D \rightarrow M$  are analytic curves, and  $D' \subseteq D$  an interval, then

$$\gamma|_{D'} = \gamma'|_{D'} \implies \gamma = \gamma'.$$

PROOF: Let  $A \subseteq D$  denote the union of all intervals  $A'$  with  $D' \subseteq A' \subseteq D$  and  $\gamma|_{A'} = \gamma'|_{A'}$ . Then,  $A$  is closed in  $D$  by continuity, as well as open in  $D$  by analyticity of  $\gamma$  and  $\gamma'$ . ■

Next, let us show that

#### Lemma 2.2

Let  $\gamma: I \rightarrow M$  be an analytic embedding, and  $\gamma': I' \rightarrow M$  an analytic curve. If there are sequences  $I \setminus \{t\} \supseteq \{t_n\}_{n \in \mathbb{N}} \rightarrow t \in I$  and  $I' \setminus \{t'\} \supseteq \{t'_n\}_{n \in \mathbb{N}} \rightarrow t' \in I'$  with  $\gamma(t_n) = \gamma'(t'_n)$  for each  $n \in \mathbb{N}$ , then  $\gamma'|_{J'} = \gamma \circ \rho$  holds for some analytic map  $\rho: J' \rightarrow D$  with  $\rho(t') = t$ , for intervals  $J', D$  with  $J'$  open.

PROOF: Let  $(U, \psi)$  be an analytic submanifold chart of  $\text{im}[\gamma]$  which is centred at  $x := \gamma(t) = \gamma'(t')$ , and maps  $\text{im}[\gamma] \cap U$  into the  $x_1$ -axis. We choose an open interval  $J' \subseteq I'$  with  $t' \in J'$  and  $\gamma'(J') \subseteq U$ , and consider the analytic functions  $f_k := \psi^k \circ \gamma'|_{J'}$  for  $k = 2, \dots, \dim[M]$ . Then,  $t'$  is an accumulation point of zeroes of each  $f_k$ , so that  $f_k = 0$  holds by analyticity. Thus, we have  $\psi(\gamma'(J')) \subseteq \psi(\text{im}[\gamma] \cap U)$ , and since  $\gamma^{-1} \circ \psi^{-1}|_{\psi(U \cap \text{im}[\gamma])}$  and  $\psi \circ \gamma'|_{J'}$  are analytic, we can just define  $\rho := \gamma^{-1} \circ \gamma'|_{J'}$ . ■

From this, we immediately obtain

**Lemma 2.3**

Let  $\gamma: I \rightarrow M$ ,  $\gamma': I' \rightarrow M$  be analytic embeddings, and  $x$  an accumulation point of  $\text{im}[\gamma] \cap \text{im}[\gamma']$  (w.r.t. the subspace topology inherited from  $M$ ). Then,  $\gamma(J) = \gamma'(J')$  holds for some open intervals  $J \subseteq I$  and  $J' \subseteq I'$  with  $x \in \gamma(J) \cap \gamma'(J')$ .

Here, and in the following, by an accumulation point of a topological space  $X$ , we will understand an element  $x \in X$ , for which we find a net  $\{x_\alpha\}_{\alpha \in I} \subseteq X \setminus \{x\}$  with  $\lim_\alpha x_\alpha = x$ .

The above lemma, then will often be used in combination with

**Lemma 2.4**

Let  $M$  be an analytic manifold, and  $\gamma: D \rightarrow M$ ,  $\gamma': D' \rightarrow M$  analytic embeddings with  $\gamma(D) = \gamma'(D')$ . Then,  $\gamma = \gamma' \circ \rho$  holds for some (necessarily unique) analytic diffeomorphism  $\rho: D \rightarrow D'$ .

PROOF: Let  $\tilde{\gamma}: I \rightarrow M$  be an analytic embedding extending  $\gamma'$ . Then, since  $(I, \tilde{\gamma})$  is an embedded analytic submanifold, we just have  $\rho = \tilde{\gamma}'^{-1} \circ \gamma$ . ■

Next, an analytic (immersive) curve  $\gamma: D \rightarrow M$  is said to be **maximal** iff it has no proper extension, i.e., iff  $\tilde{\gamma} = \gamma$  holds for each analytic (immersive) extension  $\tilde{\gamma}$  of  $\gamma$ ; analogous conventions will hold for analytic maps and diffeomorphisms  $\rho: D \rightarrow D'$ . Observe that each such maximal  $\gamma$  or  $\rho$  necessarily has open domain, and we conclude from Lemma 2.1 that

**Lemma 2.5**

Each analytic (immersive) curve admits a unique maximal analytic (immersive) extension.

PROOF: Let  $\gamma: D \rightarrow M$  be an analytic curve, denote by  $E$  the set of all analytic (immersive) extensions of  $\gamma$  defined on an open interval, and define its maximal analytic (immersive) extension

$$\bar{\gamma}: I := \bigcup_{\delta \in E} \text{dom}[\delta] \rightarrow M \quad \text{by} \quad \bar{\gamma}(t) := \delta(t) \quad \text{for} \quad \delta \in E \quad \text{with} \quad t \in \text{dom}[\delta].$$

Then,  $\bar{\gamma}$  is well defined by Lemma 2.1, because if  $\delta' \in E$  is another extension with  $t \in \text{dom}[\delta']$ , we have  $D \subseteq \text{dom}[\delta] \cap \text{dom}[\delta']$ , whereby the right hand side is an interval containing  $t$ . Finally,  $\bar{\gamma}$  is maximal, because for each  $\xi \in E$ , we have  $\text{dom}[\xi] \subseteq I$  by definition. In particular, if  $\xi$  is maximal, we must have  $\text{dom}[\xi] = I$ , hence  $\xi = \bar{\gamma}$  by Lemma 2.1. ■

Then, for  $\gamma$  an analytic (immersive) curve,  $\bar{\gamma}$  will always denote its maximal analytic (immersive) extension. Similarly, if  $\rho: I \rightarrow I'$  is an analytic map,  $\bar{\rho}$  will always denote its maximal analytic extension, as well as its maximal analytic immersive extension iff  $\rho$  is a diffeomorphism.

Let us finally show that

**Lemma 2.6**

Let  $\varphi: G \times M \rightarrow M$  be a left action,  $\gamma: I \rightarrow M$  an analytic curve, and  $\{g_n\}_{n \in \mathbb{N}} \subseteq G \setminus G_x$ ,  $\{t_n\}_{n \in \mathbb{N}} \subseteq I \setminus \{t\}$  sequences with  $\lim_n g_n = e$  and  $\lim_n t_n = t \in I$ . Then, if  $g_n \cdot \gamma(t) = \gamma(t_n)$  holds for each  $n \in \mathbb{N}$ , we have  $\gamma(J) \subseteq G \cdot \gamma(t)$  for some open interval  $J$  containing  $t$ .

PROOF: Write  $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}_x$  for  $x := \gamma(t)$ , and fix open neighbourhoods  $U \subseteq \mathfrak{c}$  and  $V \subseteq \mathfrak{g}_x$  of zero, such that

$$h: U \times V \rightarrow W, \quad (u, v) \mapsto \exp(u) \cdot \exp(v)$$

is an analytic diffeomorphism to an open neighbourhood  $W \subseteq G$  of the identity  $e \in G$ . Then,

- ▷ Since the differential  $d_e \iota = d_e \varphi_x|_{TU}$  at  $e$  of  $\iota := \varphi_x \circ \exp|_U$  is injective, shrinking  $U$  if necessary, we can assume that  $(U, \iota)$  is an embedded analytic submanifold.
- ▷ We have  $\{g_n\}_{n \geq p} \subseteq W$  for some  $p \in \mathbb{N}$ , hence  $g_n = h(u_n, v_n)$  for  $(u_n, v_n) \in U \times V$  unique, for each  $n \geq p$ . Moreover, we find  $p' \geq p$ , such that  $\{g_n \cdot x\}_{n \geq p'}$  is contained in some analytic submanifold chart  $(O, \psi)$  of  $(U, \iota)$  with  $\psi(x) = 0$ .

▷ Then,  $\{g_n \cdot x\}_{n \geq p'}$  is contained in  $O \cap \iota(U)$ , because

$$g_n \cdot x = \exp(u_n) \cdot \exp(v_n) \cdot x = \exp(u_n) \cdot x = \iota(u_n) \in \iota(U) \quad \forall n \geq p'. \quad (4)$$

▷ Let  $\psi: O \rightarrow O' \subseteq \mathbb{R}^n$  map  $O \cap \iota(U)$  into  $\{0\} \times \mathbb{R}^{\dim[M]-m}$ , and let  $J \subseteq I$  be an open interval with  $t \in J$  and  $\gamma(J) \subseteq O$ . Then, 0 is an accumulation point of zeroes of the analytic functions  $f_k := \psi^k \circ \gamma|_J$  for  $k = m+1, \dots, n$ , because

$$f_k(t_n) = \psi^k(g_n \cdot x) \stackrel{(4)}{=} (\psi^k \circ \iota)(u_n) = 0 \quad \forall n \geq p'.$$

Thus,  $f_k = 0$  holds for  $k = m+1, \dots, n$  by analyticity, hence  $\gamma(J) \subseteq \iota(U) \subseteq G \cdot x$ . ■

### 2.2.2 Relations between curves

Let  $\gamma: D \rightarrow M$  and  $\gamma': D' \rightarrow M$  be two analytic immersions. Then,

- We will write  $\gamma \sim_o \gamma'$  iff  $\gamma(J) = \gamma'(J')$  holds for open intervals  $J \subseteq D$  and  $J' \subseteq D'$ , on which  $\gamma$  and  $\gamma'$  are embeddings, respectively.

Observe that then by Lemma 2.4,  $\gamma = \gamma' \circ \rho$  holds for some unique analytic diffeomorphism  $\rho: J \rightarrow J'$ .

- Moreover, if it is necessary to be more precise, we will say that  $\gamma \sim_o \gamma'$  holds w.r.t.  $\rho$  iff  $\rho: J \rightarrow J'$  is an analytic diffeomorphism with  $\gamma|_J = \gamma' \circ \rho$ , such that  $\gamma|_J$  and  $\gamma|_{J'}$  are embeddings.

Then, if  $\varphi: G \times M \rightarrow M$  is analytic in  $M$ , we obviously have

$$g \cdot \gamma \sim_o g \cdot \gamma' \quad \Longleftrightarrow \quad \gamma \sim_o \gamma' \quad \Longleftrightarrow \quad \gamma \sim_o \gamma' \circ \tau$$

for each  $g \in G$ , and each analytic diffeomorphism  $\tau: D'' \rightarrow D'$ .

Now, assume that  $\gamma \sim_o \gamma'$  holds w.r.t.  $\rho$ , i.e., that  $\gamma|_J = \gamma' \circ \rho$  holds for some analytic diffeomorphism  $\rho: J \rightarrow J'$ . We next want to figure out, what might happen, if we try to extend such a relation to the whole domain of  $\gamma$ . For this, observe that

$$\gamma|_C = \gamma' \circ \bar{\rho}|_C \quad \text{holds on the interval} \quad C := D \cap \bar{\rho}^{-1}(D')$$

by Lemma 2.1, and that  $C$  is maximal w.r.t. this property. Then, for  $D = K$  and  $D' = K'$  compact, we obtain

#### Lemma 2.7

Let  $\gamma: K \rightarrow M$  and  $\gamma': K' \rightarrow M$  be analytic immersions with  $\gamma|_B = \gamma' \circ \rho$  for some analytic diffeomorphism  $\rho: B \rightarrow B'$ . Then,  $C := K \cap \bar{\rho}^{-1}(K')$  is a compact interval, and we have

$$a < c' \quad \implies \quad \bar{\rho}(c') \in \{a', b'\} \quad \text{as well as} \quad c < b \quad \implies \quad \bar{\rho}(c) \in \{a', b'\} \quad (5)$$

for  $C = [c', c]$ ,  $K = [a, b]$ , and  $K' = [a', b']$ .

PROOF: Let  $t$  be contained in the closure  $\bar{C} \subseteq K$  of  $C$ , and let  $\{t_n\}_{n \in \mathbb{N}} \subseteq C \setminus \{t\}$  converge to  $t$ . Since  $\{\bar{\rho}(t_n)\}_{n \in \mathbb{N}} \subseteq K'$  holds, by compactness of  $K'$ , we can assume that  $\lim_n \bar{\rho}(t_n) = t' \in K'$  exists. Then, we find open intervals  $I, I'$  with  $t \in I$ ,  $t' \in I'$ , such that  $\bar{\gamma}|_I, \bar{\gamma}'|_{I'}$  are embeddings, so that

▷ Combining Lemma 2.3 with Lemma 2.4, and shrinking  $I, I'$  if necessary, we find a unique analytic diffeomorphism  $\tau: I \rightarrow I'$  with  $\bar{\gamma}|_I = \bar{\gamma}' \circ \tau$ .

▷ Since  $\bar{\rho}$  is monotonous, we find an open interval  $J \subseteq I$  containing  $t$ , such that  $\bar{\rho}(C \cap J) \subseteq I'$  holds; and then  $\tau$  coincides with  $\bar{\rho}$  on  $C \cap J$ , because

$$\bar{\gamma}|_{C \cap J} = \gamma|_{C \cap J} = \gamma' \circ \bar{\rho}|_{C \cap J} = \bar{\gamma}' \circ \bar{\rho}|_{C \cap J} \quad \implies \quad \tau|_{C \cap J} = \bar{\rho}|_{C \cap J}$$

by injectivity of  $\bar{\gamma}'$  on  $I'$ .



▷ Thus, by maximality,  $\bar{\rho}$  is defined on an open interval around each  $t \in \bar{C}$ .

In particular,  $C = [c', c]$  is compact, because  $\bar{\rho}(C) \subseteq K'$  implies  $\bar{\rho}(\bar{C}) \subseteq K'$ . Then, if  $c < b$  and  $\bar{\rho}(c) \in (a', b')$  holds, we find an open interval  $I \subseteq K$  with  $c \in I$  and  $\bar{\rho}(I) \subseteq K'$ . This, however, contradicts the definition of  $C$ ; and in the same way, we obtain a contradiction if  $a < c'$  and  $\bar{\rho}(c') \in (a', b')$  holds. ■

The above lemma provides us with the following useful corollaries:

**Corollary 2.8**

Suppose that  $\gamma|_J = \gamma' \circ \rho$  holds for analytic immersions  $\gamma: [a, b] \rightarrow M$ ,  $\gamma': K' \rightarrow M$ , and an analytic diffeomorphism  $\rho: J \rightarrow J'$ . Then, for each  $t \in J$ , we have

$$\text{im}[\gamma'] \not\subseteq \text{im}[\gamma] \quad \implies \quad \gamma([a, t]) \subseteq \text{im}[\gamma'] \quad \text{or} \quad \gamma([t, b]) \subseteq \text{im}[\gamma'].$$

PROOF: Let  $[c', c] = C := [a, b] \cap \bar{\rho}^{-1}(K')$  be as in Lemma 2.7. Then, since  $\text{im}[\gamma'] \not\subseteq \text{im}[\gamma]$  holds, we cannot have  $\bar{\rho}(C) = K'$ , so that  $c' = a$  or  $c = b$  must hold by (5). Thus, the claim is clear from

$$\begin{aligned} c' = a &\implies [a, t] \subseteq C \implies \gamma([a, t]) = \gamma'(\bar{\rho}([a, t])) \subseteq \text{im}[\gamma'] \\ c = b &\implies [t, b] \subseteq C \implies \gamma([t, b]) = \gamma'(\bar{\rho}([t, b])) \subseteq \text{im}[\gamma']. \end{aligned} \quad \blacksquare$$

**Corollary 2.9**

If  $\gamma: [a', a] \rightarrow M$  is an analytic immersion, we cannot have  $\gamma|_{[a', r]} = \gamma \circ \rho$  for some negative analytic diffeomorphism  $\rho: [a', r] \rightarrow [s, a]$  with  $r, s \in (a', a)$ .

PROOF: If  $\rho$  is such a negative diffeomorphism, we have  $[a', a] \cap \bar{\rho}^{-1}([a', a]) = [a', c]$  for some  $c \in [r, a]$ . Thus,

▷ If  $c = a$  holds, we have  $\gamma = \gamma \circ \bar{\rho}|_{[a', a]}$ , as well as  $\bar{\rho}(\tau) = \tau$  for

$$\tau := \sup\{t \in [a', a] : t \leq \bar{\rho}(t)\} \in (a', a)$$

by continuity, because  $a' < a = \bar{\rho}(a')$  holds by negativity, hence  $a > \bar{\rho}(a)$  by injectivity. Since  $\gamma$  is injective on a neighbourhood of  $\tau$ , this contradicts negativity of  $\bar{\rho}$ .

▷ If  $c < a$  holds, we have  $\bar{\rho}(c) = a'$  by (5), because  $\bar{\rho}(a') = a$  holds. Thus, we have  $\gamma = \gamma \circ \bar{\rho}^{-1}|_{[a', a]}$ , as well as  $\bar{\rho}^{-1}(\tau) = \tau$  for

$$\tau := \inf\{t \in [a', a] : \bar{\rho}^{-1}(t) \leq t\} \in (a', a)$$

by continuity, because  $\bar{\rho}^{-1}(a) = a' < a$  and  $\bar{\rho}^{-1}(a') = c > a'$  holds. Since  $\gamma$  is injective on a neighbourhood of  $\tau$ , this contradicts negativity of  $\bar{\rho}^{-1}$ . ■

### 2.2.3 Self relations of curves

We will say that the analytic immersion  $\gamma: D \rightarrow M$  is **self related** iff  $\gamma(K) = \gamma(K')$  holds for disjoint compact intervals  $K, K' \subseteq D$ , on which  $\gamma$  is an embedding. Let us first show that

**Lemma 2.10**

If  $\gamma: K \rightarrow M$  is an immersion, then  $\gamma = \gamma \circ \rho$  cannot hold for some homeomorphism  $\rho: K \rightarrow K' \subset K$ .

PROOF: If  $\rho$  is such a homeomorphism, we can define  $K_n := \rho^n(K)$  for each  $n \in \mathbb{N}$ ,<sup>8</sup> with what  $K_{n+1} \subset K_n$  holds for each  $n \in \mathbb{N}$ . We fix  $k \in K \setminus K'$ , and define  $\{k_n\}_{n \in \mathbb{N}} \subseteq K$  by  $k_n := \rho^n(k)$  for each  $n \in \mathbb{N}$ . Then,

▷ The  $k_n$  are mutually different, because the sets  $\rho^n(K \setminus K') = K_n \setminus K_{n+1}$  are mutually disjoint.

▷ Since  $K$  is compact, and  $\{k_n\}_{n \in \mathbb{N}} \subseteq K$  holds, we find  $\phi: \mathbb{N} \rightarrow \mathbb{N}$  injective and increasing, such that  $\lim_n k_{\phi(n)} = k' \in K$  exists.

---

<sup>8</sup>Of course, here  $\rho^n$  means to apply  $\rho$   $n$ -times.

Then, by construction, for each  $n \in \mathbb{N}$ , we have  $\gamma(k_{n+1}) = \gamma(\rho(k_n)) = \gamma(k_n)$ , hence  $\gamma(k_n) = \gamma(k_0)$  for all  $n \in \mathbb{N}$ . Thus,  $\gamma(k_{\phi(n)}) = \gamma(k_0)$  holds for all  $n \in \mathbb{N}$ , which contradicts that  $\gamma$  is injective on some neighbourhood of  $k'$ . ■

From this, we easily obtain

**Lemma 2.11**

Let  $\gamma: I \rightarrow M$  be an analytic immersion, and  $D \subset I$  an interval. If  $\gamma|_D$  is self related, we have  $\gamma(J) = \gamma(J')$  for open intervals  $J \subseteq D$  and  $J' \subseteq I \setminus D$  on which  $\gamma$  is an embedding.

PROOF: By assumption, we find an analytic diffeomorphism  $\rho: D \supseteq [a', r] \rightarrow [s, a] \subseteq D$  for  $r < s$  with  $\gamma|_{[a', r]} = \gamma \circ \rho$ . Then,  $\dot{\rho} > 0$  holds by Corollary 2.9 (applied to  $\gamma|_{[a', a]}$ ), hence  $\rho(a') = s$  and  $\rho(r) = a$ .

Since  $D$  is properly contained in  $I$ ,  $\sup(D)$  or  $\inf(D)$  must exist in  $I$ . In the first case (the second case follows analogously), we have  $t + \epsilon \in I$  for  $t := \sup[D]$ , and some  $\epsilon > 0$ . Then,  $[a', c] = [a', t] \cap \bar{\rho}^{-1}([s, t + \epsilon])$  holds for some  $r \leq c \leq t$ , and we conclude that

- ▷ If  $\bar{\rho}(c) > t$  holds, the claim is clear from  $\gamma|_{[a', c]} = \gamma \circ \bar{\rho}|_{[a', c]}$ , as we have  $[a', c] \subseteq D$  and  $[t, t + \epsilon] \subseteq I \setminus D$ .
- ▷ If  $\bar{\rho}(c) \leq t$ , holds we must have  $c = t$ , because  $c < t$  implies  $\bar{\rho}(c) = t + \epsilon > t$  by (5). Consequently, we have  $\bar{\rho}(t) = \bar{\rho}(c) \leq t$  for  $c = t$ ,

$$\text{hence } \gamma|_{[a', t]} = \gamma|_{[a', t]} \circ \tau \quad \text{for } \tau := \bar{\rho}|_{[a', t]}: [a', t] \rightarrow [s, \bar{\rho}(t)] \subset [a', t],$$

which contradicts Lemma 2.10. ■

Finally, let us provide the following conditions for self relatedness of curves.

**Lemma 2.12**

Let  $\gamma: D \rightarrow M$  and  $\gamma': D' \rightarrow M$  be analytic immersions with

$$\begin{aligned} \gamma|_B &= \gamma' \circ \phi & \text{for } \phi: B \rightarrow D' & \text{an analytic diffeomorphism,} \\ \gamma|_J &= \gamma' \circ \psi & \text{for } \psi: J \rightarrow J' & \text{an analytic diffeomorphism.} \end{aligned}$$

Then,  $\gamma$  is self related, if  $J \not\subseteq B$  or  $J \subseteq B$  and  $\phi|_J \neq \psi$  holds.

PROOF: Let  $L \subseteq J'$  be such that  $\gamma'|_L$  is an embedding, and define  $B \supseteq K := \phi^{-1}(L)$ , as well as  $J \supseteq K' := \psi^{-1}(L)$ . Then,  $\gamma|_K$  and  $\gamma|_{K'}$  are embeddings with  $\gamma(K) = \gamma'(L) = \gamma(K')$ , and we conclude that

- ▷ If  $J \not\subseteq B$  holds, then  $J \setminus B$  has non-empty interior. Thus, shrinking  $J$  if necessary, we can assume that  $J \cap B = \emptyset$  holds right from the beginning, with what  $\gamma$  is self related, because then  $K \cap K' = \emptyset$  holds.
- ▷ If  $J \subseteq B$  and  $\phi|_J \neq \psi$  holds, we can shrink  $L$  in such a way that  $K \cap K' = \emptyset$  holds as well.

For this, assume that the statement is wrong, and observe that for each  $t \in L$ , we find a decreasing sequence  $\{L_n\}_{n \in \mathbb{N}}$  of compact intervals contained in  $L$  with  $\bigcap_n L_n = \{t\}$ . Then, by assumption,  $\phi^{-1}(L_n) \cap \psi^{-1}(L_n) \neq \emptyset$  holds for each  $n \in \mathbb{N}$ , so that we find  $\{r_n\}_{n \in \mathbb{N}}, \{s_n\}_{n \in \mathbb{N}} \subseteq L$  with  $\lim_n r_n = t = \lim_n s_n$ , as well as  $\phi^{-1}(r_n) = \psi^{-1}(s_n)$  for each  $n \in \mathbb{N}$ , hence

$$\phi^{-1}(t) = \lim_n \phi^{-1}(r_n) = \lim_n \psi^{-1}(s_n) = \psi^{-1}(t).$$

Since, this holds for each  $t \in L$ , we have  $\phi^{-1}|_L = \psi^{-1}|_L$ , hence  $\phi^{-1}|_{J'} = \psi^{-1}|_{J'}$  by Lemma 2.1, so that  $\phi|_J = \psi$  contradicts the assumptions. ■

## 2.3 Reparametrizations

In this subsection, we will provide some further statements concerning reparametrizations of analytic immersive curves, to be used in Section 4. Indeed, the arguments from Lemma 2.7 also work for non-compact domains, and to figure out the possible cases efficiently, let us write  $\gamma \rightsquigarrow_{t, t'} \gamma'$  for analytic immersions  $\gamma: D \rightarrow M$ ,  $\gamma': D' \rightarrow M$  with  $t \in D$ ,  $t' \in D'$  iff there is some analytic diffeomorphism  $\rho: D \rightarrow D'$  with  $\rho(t) = t'$  and  $\gamma = \gamma' \circ \rho$ . This diffeomorphism is uniquely determined, because

$$\gamma|_A \rightsquigarrow_{t, t'} \gamma'|_{A'} \quad \text{w.r.t. } \tau: A \rightarrow A' \quad \implies \quad \tau|_{A \cap D} = \rho|_{A \cap D}. \quad (6)$$

In fact,

- ▷ Combining  $\gamma \rightsquigarrow_{t,t'} \gamma'$  with Lemma 2.3, we find open intervals  $I, I'$  with  $t \in I, t' \in I'$  and  $\bar{\gamma}(I) = \bar{\gamma}'(I')$ , such that  $\bar{\gamma}|_I$  and  $\bar{\gamma}'|_{I'}$  are embeddings.
- ▷ Then,  $\bar{\gamma}|_I = \bar{\gamma}' \circ \mu$  holds for an analytic diffeomorphism, whereby we have  $\rho(J \cap D), \tau(J \cap D) \subseteq I'$  for some open interval  $J \subseteq I$  containing  $t$ , by continuity of  $\rho$  and  $\tau$ . Then, injectivity of  $\bar{\gamma}'$  on  $I'$  shows (cf. proof of Lemma 2.7) that

$$\tau|_{J \cap A \cap D} = \mu|_{J \cap A \cap D} = \rho|_{J \cap A \cap D} \implies \tau|_{A \cap D} = \rho|_{A \cap D}.$$

Here, the implication is trivial if  $A \cap D = \{t\}$  holds, and follows from Lemma 2.1 in the other case.

Now, if  $\gamma: D \rightarrow M$  and  $\gamma': D' \rightarrow M$  are as above, and  $\gamma|_K = \gamma' \circ \rho$  holds for some analytic diffeomorphism  $\rho: K \rightarrow K'$ , we can fix some  $t \in \text{int}[K]$ , and define  $t' := \rho(t)$ . Then, the question how  $C := D \cap \bar{\rho}^{-1}(D')$  and  $C' := \bar{\rho}(C)$  look like, can be formulated in terms of the relation, we have introduced above.

For this, let us define  $K_+ := K \cap [t, \infty)$  and  $K_- := K \cap (-\infty, t]$ , as well as  $K'_+ := K' \cap [t', \infty)$  and  $K'_- := K' \cap (-\infty, t']$ . Then, we have

$$\begin{aligned} \dot{\bar{\rho}} > 0 &\iff \gamma|_{K_+} \rightsquigarrow_{t,t'} \gamma'|_{K'_+} && \text{and} && \gamma|_{K_-} \rightsquigarrow_{t,t'} \gamma'|_{K'_-} \\ \dot{\bar{\rho}} < 0 &\iff \gamma|_{K_+} \rightsquigarrow_{t,t'} \gamma'|_{K'_-} && \text{and} && \gamma|_{K_-} \rightsquigarrow_{t,t'} \gamma'|_{K'_+}. \end{aligned}$$

Next, let us split the intervals  $D, D', C, C'$  into their positive and negative parts as well, i.e., let

- $D_+, C_+$  and  $D_-, C_-$  denote the intersections of  $D, C$  with  $[t, \infty)$  and  $(-\infty, t]$ , respectively,
- $D'_+, C'_+$  and  $D'_-, C'_-$  denote the intersections of  $D', C'$  with  $[t', \infty)$  and  $(-\infty, t']$ , respectively.

Then,  $\bar{\rho}: C_{\pm} \rightarrow C'_{\pm}$  holds iff  $\dot{\bar{\rho}} > 0$ , and  $\bar{\rho}: C_{\pm} \rightarrow C'_{\mp}$  holds iff  $\dot{\bar{\rho}} < 0$ , so that<sup>9</sup>

- If  $\dot{\bar{\rho}} > 0$ , then  $C_{+/-} = D_{+/-} \iff \gamma|_{D_{+/-}} \rightsquigarrow_{t,t'} \gamma'|_{C'_{+/-}}.$
- If  $\dot{\bar{\rho}} < 0$ , then  $C_{+/-} = D_{+/-} \iff \gamma|_{D_{+/-}} \rightsquigarrow_{t,t'} \gamma'|_{C'_{-/+}}.$

Thus, it only remains to investigate what might happen if  $C_+ \subset D_+$  or  $C_- \subset D_-$  holds.

### Lemma 2.13

Let  $C_{+/-} \subset D_{+/-}$ . Then,

- If  $\dot{\bar{\rho}} > 0$ , we have  $C'_{+/-} = D'_{+/-}$ , hence  $\gamma|_{C_{+/-}} \rightsquigarrow_{t,t'} \gamma'|_{D'_{+/-}}.$
- If  $\dot{\bar{\rho}} < 0$ , we have  $C'_{+/-} = D'_{-/+}$ , hence  $\gamma|_{C_{+/-}} \rightsquigarrow_{t,t'} \gamma'|_{D'_{-/+}}.$

PROOF: We only show the case where  $C_+ \subset D_+$  and  $\dot{\bar{\rho}} > 0$  holds, because the other cases follow analogously.

Now, first observe that we either have  $C_+ = [t, c]$  and  $C'_+ = [t', c']$  or  $C_+ = [t, c)$  and  $C'_+ = [t', c')$  for some  $c > t$  and  $c' > t'$ . In any case,  $\bar{\gamma}$  is defined on an open interval  $I$  containing  $c$ , on which it is an embedding, just because  $C_+ \subset D_+$  holds. Moreover, if the statement is wrong,  $\bar{\gamma}'$  is defined on some open interval  $I'$  containing  $c'$ , on which it is an embedding. Then, by Lemma 2.3 we can shrink  $I$  and  $I'$  in such a way that  $\bar{\gamma}(I) = \bar{\gamma}'(I')$  holds,<sup>10</sup> and by the same arguments as in Lemma 2.7, we see that  $\bar{\rho}$  is defined on  $C_+ \cup I$ . Thus,

- ▷ If  $C_+ = [t, c]$  holds, we have  $C'_+ = [t', c']$ , and since  $C_+ \subset D_+, C'_+ \subset D'_+$  holds, we find  $\epsilon > 0$  with  $[t, c + \epsilon) \subseteq D_+$  and  $\bar{\rho}([t, c + \epsilon)) \subseteq D'_+$ . This, however, contradicts the definition of  $C$ .
- ▷ If  $C_+ = [t, c)$ , hence  $C'_+ = [t', c')$  holds, we have  $[t, c] \subseteq D_+$  and  $\bar{\rho}([t, c]) = [t', c'] \subseteq D'_+$ , which contradicts the definition of  $C$  as well. ■

Let us finally provide some notations, which are adapted to the situation in Section 4. There, we will be concerned with restrictions of curves to compact and half-open intervals.

<sup>9</sup>More precisely, in the first point (and analogously for the second point and Lemma 2.13),  $+/-$  means that we have  $C_+ = D_+$  iff  $\gamma|_{D_+} \rightsquigarrow_{t,t'} \gamma'|_{C_+}$  holds, as well as  $C_- = D_-$  iff  $\gamma|_{D_-} \rightsquigarrow_{t,t'} \gamma'|_{C_-}$  holds.

<sup>10</sup>More precisely, if  $C_+$  is of the form  $[t, c)$  and  $\{t_n\}_{n \in \mathbb{N}} \subseteq [t, c)$  is monotonously increasing with limit  $c$ , then  $\{\rho(t_n)\}_{n \in \mathbb{N}} \subseteq [t', c')$  necessarily converges to  $c'$ , by positivity of  $\rho$ .

- If  $D = (i', \tau]$  and  $D' = [\tau, i)$  holds, we will write  $\gamma \rightsquigarrow \gamma'$  iff

$$\gamma \rightsquigarrow_{\tau, \tau} \gamma' \quad \text{or} \quad \gamma|_{(j', \tau]} \rightsquigarrow_{\tau, \tau} \gamma' \quad \text{for} \quad i' < j' \quad \text{or} \quad \gamma \rightsquigarrow_{\tau, \tau} \gamma'|_{[\tau, j)} \quad \text{for} \quad j < i \quad \text{holds.}$$

It is then clear from the above discussions that only one of these cases can occur, and that the respective reals  $j$  and  $j'$  are uniquely determined.

- In addition to that, we will write  $\gamma \rightsquigarrow \gamma'$ :

$$\begin{array}{llllll} \text{If} & D = [a, b], D' = [a', b'] & \text{and} & \gamma \rightsquigarrow_{a, a'} \gamma' & \text{or} & \gamma \rightsquigarrow_{a, b'} \gamma' & \text{holds.}^{11} \\ \text{If} & D = [a, b], D' = [a', b') & \text{and} & \gamma|_{[a, j]} \rightsquigarrow_{a, a'} \gamma' & \text{or} & \gamma|_{(j', b]} \rightsquigarrow_{b, a'} \gamma' & \text{holds.} \\ \text{If} & D = [a, b], D' = (a', b'] & \text{and} & \gamma|_{(j', b]} \rightsquigarrow_{b, b'} \gamma' & \text{or} & \gamma|_{[a, j)} \rightsquigarrow_{a, b'} \gamma' & \text{holds.} \end{array}$$

Again, in each of the above cases, only one of the mentioned situations can hold. This now follows from Corollary 2.9; and uniqueness of the respective reals  $j$  and  $j'$  follows as above. Then, instead of  $\gamma \rightsquigarrow \gamma'$ , we will also write  $\gamma \rightsquigarrow_+ \gamma'$  or  $\gamma \rightsquigarrow_- \gamma'$  iff one of the above cases on the left or on the right hand side holds, respectively.

## 2.4 Regularity and stabilizers

This subsection collects the definitions and facts concerning group actions that we will need in the following.

### 2.4.1 Regularity

Let us start with the following

#### Definition 2.14 (Regularity)

Let  $\varphi: G \times M \rightarrow M$  be some fixed left action. We will say that

- $x \in M$  is **sated** iff there are no two different  $y, z \in M \setminus \{x\}$  with (this is equivalent to i) in Section 1)

$$\lim_n g_n \cdot y = x = \lim_n g_n \cdot z \quad \text{for some sequence} \quad \{g_n\}_{n \in \mathbb{N}} \subseteq G.$$

- $x$  is **stable** iff  $\lim_n g_n \cdot x = x$  for  $\{g_n\}_{n \in \mathbb{N}} \subseteq G \setminus G_x$  implies that  $\{h_n \cdot g_n \cdot h'_n\}_{n \in \mathbb{N}}$  has a convergent subsequence<sup>12</sup> for some sequences  $\{h_n\}_{n \in \mathbb{N}}, \{h'_n\}_{n \in \mathbb{N}} \subseteq G_{[x]}$ .
- $x$  is **regular** iff it is **sated** and **stable**.
- $\varphi$  is **regular/sated/stable**, iff each  $x \in M$  is **regular/sated/stable**.

#### Remark 2.15

- 1) The point  $x \in M$  is **regular/sated/stable** iff each  $y \in G \cdot x$  is **regular/sated/stable**.

This is straightforward for satedness; and for stability, one can use that  $G_{[x]} = g \cdot G_{[x]} \cdot g^{-1}$  holds for all  $g \in G$ , and that  $G_y = g \cdot G_x \cdot g^{-1}$  holds for  $y \in M$  and  $g \in G$  with  $y = g \cdot x$ .

- 2) The action  $\varphi$  is **sated**,

- a) If there is some  $G$ -invariant continuous metric  $d$  on  $M$ , because then  $\lim_n g_n \cdot y = x = \lim_n g_n \cdot z$  implies  $0 = d(x, x) = \lim_n d(g_n \cdot y, g_n \cdot z) = d(y, z)$ .
- b) If  $M$  is a topological group, such that  $\varphi(g, x) = \phi(g) \cdot x$  holds for each  $g \in G$ , and each  $x \in M$ , for some continuous group homomorphism  $\phi: G \rightarrow M$ .

In fact, then  $\lim_n \phi(g_n) \cdot y = x$  implies  $\lim_n \phi(g_n) = h := x \cdot y^{-1}$ , so that

$$\lim_n \phi(g_n) \cdot z = x \quad \implies \quad h \cdot z = x \quad \implies \quad z = y.$$

In addition to that, we have  $G_x = G_{[x]} = \ker[\phi]$  for each  $x \in M$ .

<sup>11</sup>Obviously, the first case is equivalent to  $\gamma \rightsquigarrow_{b, b'} \gamma'$ , and the second one to  $\gamma \rightsquigarrow_{b, a'} \gamma'$ .

<sup>12</sup>This subsequence necessarily converges to some element in  $G_x$ .

Moreover, in the situation of b),

- $\varphi$  is **stable** iff it is stable at  $e_M \in M$ , because

$$\lim_n g_n \cdot x = x \implies \lim_n g_n \cdot e_M = e_M \quad \text{as well as} \quad G_{[x]} = G_x = \ker[\phi] = G_{e_M} = G_{[e_M]}$$

holds for each  $x \in M$ .

- $\varphi$  is **stable** if  $\phi \circ s = \text{id}_V$  holds for some continuous map<sup>13</sup>  $s: V := U \cap \phi(G) \rightarrow G$ , with  $U$  a neighbourhood of  $e_M$ . In fact, then  $\lim_n \phi(g_n) \cdot x = x$  implies  $\lim_n \phi(g_n) = e_M$ , so that for  $n$  such large that  $\phi(g_n) \in U$  holds, we have

$$\begin{aligned} \lim_n \phi(g_n) \cdot x = x &\implies \lim_n \phi(g_n) = e_M \\ &\implies \lim_n s(\phi(g_n)) = s(\phi(e)) \implies \lim_n g_n \cdot h'_n = s(\phi(e)) \end{aligned}$$

for  $h'_n := g_n^{-1} \cdot s(\phi(g_n)) \in \ker[\phi] = G_{[x]}$ . In particular, each closed subgroup of a Lie group  $G$  acts via left multiplication **regularly** on  $G$ .

- 3) The point  $x \in M$  is **regular** if  $\varphi_x$  is proper.<sup>14</sup> In fact, then  $x$  is obviously stable; and if  $\lim_n g_n \cdot y = x = \lim_n g_n \cdot z$  holds, then  $\lim_n g_n$  can be assumed to exist, with what  $y = z$  follows. Thus, pointwise proper, hence proper actions are **regular**. However, in general, pointwise properness is a stronger condition than regularity, as, e.g., an action cannot be pointwise proper if  $G_x$  is non-compact for some  $x \in M$ , see also Example 2.16.2.

- In fact, let  $G$  be a Lie group with closed normal (non-compact) subgroup  $H$ . Moreover, let  $\varphi$  act on  $M := G/H$  in the canonical way, i.e., via  $\varphi: (g, x) \mapsto [g \cdot x]$ . Since  $H$  is normal,  $M$  is a Lie group, and the projection  $\pi: G \rightarrow M$  is a Lie group homomorphism. Moreover, by general theory, there exists some local section  $s: U \rightarrow G$  with  $\pi \circ s = \text{id}_U$ , for  $U$  some open neighbourhood of  $[e]$ . Thus,  $\varphi$  is regular by Part 2).
- For instance, we can choose,  $G = \mathbb{R}^n$  and  $H = \mathbb{Z}^n$  (hence  $M = \mathbb{T}^n$ ), or  $G = \mathbb{R}^n$  and  $H \subseteq \mathbb{R}^n$  some  $m$ -dimensional linear subspace for  $m > 0$  (hence  $M \cong \mathbb{R}^{n-m}$ ), in order to obtain regular actions that admit non-compact stabilizers.

### Example 2.16

- 1) The origin is stable but not sated w.r.t. the multiplicative action of  $\mathbb{R}_{>0}$  on  $\mathbb{R}^n$ , but each point in  $\mathbb{R}^n \setminus \{0\}$  is regular.
- 2) For  $\lambda \in \mathbb{R}$ , the diagonal action of  $\mathbb{R}$  on the 2-Torus  $\mathbb{T}^2 = U(1) \times U(1)$

$$\varphi: \mathbb{R} \times \mathbb{T}^2 \rightarrow \mathbb{T}^2, \quad (t, (u_1, u_2)) \mapsto (e^{2\pi t \cdot i} \cdot u_1, e^{2\pi t \lambda \cdot i} \cdot u_2)$$

is sated by Remark 2.15.2, because  $\varphi(t, u) = \phi(t) \cdot u$  holds for  $\phi(t) := (e^{2\pi t \cdot i}, e^{2\pi t \lambda \cdot i})$ . Then,  $\varphi$  is stable iff  $\lambda$  is rational, because

- If  $\lambda$  is rational, then  $\ker[\phi] = \{k \cdot m \mid k \in \mathbb{Z}\}$  holds for  $m \in \mathbb{N}_{>0}$  minimal with  $m \cdot \lambda \in \mathbb{Z}$ . Thus, if  $\lim_n g_n \cdot u = u$  holds for  $\{g_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$  and  $u \in \mathbb{T}^2$ , for each  $n \in \mathbb{N}$ , we find  $h_n \in G_{[u]} = \ker[\phi]$  with  $h_n \cdot g_n \in [0, 2\pi m]$ . Thus,  $u$  is stable, because  $\{h_n \cdot g_n\}_{n \in \mathbb{N}}$  admits a convergent subsequence by compactness of  $[0, 2\pi m]$ .
- If  $\lambda$  is irrational,  $e^{2\pi t \cdot i} = 1 = e^{2\pi t \lambda \cdot i}$  implies  $t = 0$ , so that  $\ker[\phi] = \{0\}$  holds. Moreover, we find  $\mu \in \mathbb{R}$ , such that  $1, \mu, \mu \cdot \lambda$  are  $\mathbb{Q}$ -independent,<sup>15</sup> so that  $\{u^n\}_{n \in \mathbb{Z}} \subseteq \mathbb{T}^2$  for  $u := (e^{2\pi \mu \cdot i}, e^{2\pi \mu \lambda \cdot i})$  is dense by Kronecker's theorem. Then, each open neighbourhood  $U$  of  $e$  in  $\mathbb{T}^2$  contains infinitely many  $u^n$ . Thus, choosing a countable base of neighbourhoods of  $e$ , we inductively find  $\iota: \mathbb{N} \rightarrow \mathbb{Z}$  injective with  $\lim_n u^{\iota(n)} = e$  and  $|\iota(n+1)| > |\iota(n)|$  for all  $n \in \mathbb{N}$ . Now,  $u^{\iota(n)} = \phi(g_n) \cdot e$  holds for  $g_n := \iota(n) \cdot \mu \in \mathbb{R}$  for all  $n \in \mathbb{N}$ , so that we have  $\lim_n g_n \cdot e = \lim_n u^{\iota(n)} = e$ . But,  $\{g_n\}_{n \in \mathbb{N}}$  cannot admit any convergent subsequence, because  $\{|\iota(n)|\}_{n \in \mathbb{N}} \subseteq \mathbb{N}$  is strongly increasing. Thus,  $e$  cannot be stable, because  $G_{[e]} = \ker[\phi] = \{0\}$  holds.

<sup>13</sup>Of course, here continuity has to be understood w.r.t. the subspace topology inherited from  $M$  on  $V$ .

<sup>14</sup>Since manifolds are assumed to be second countable, respective subsets are compact iff they are sequentially compact. Thus, properness of  $\varphi_x$  just means that a sequence  $\{g_n\}_{n \in \mathbb{N}} \subseteq G$  admits a convergent subsequence whenever  $\lim_n \{g_n \cdot x\}_{n \in \mathbb{N}} \in M$  exists for some  $x \in M$ .

<sup>15</sup>Elsewise, for each  $\mu \in \mathbb{R}$ , we find  $q, q' \in \mathbb{Q}$  with  $\mu = \frac{q'}{1+\lambda q}$ , which contradicts that  $\mathbb{R}$  is uncountable.

### 2.4.2 Stabilizers

Let us start with the following

**Definition 2.17 (Stabilizer)**

For a curve  $\gamma: D \rightarrow M$ , we define its stabilizer subgroup by

$$G_\gamma := \{g \in G \mid g \cdot \gamma = \gamma\} = \bigcap_{t \in D} G_{\gamma(t)}.$$

Observe that  $G_\gamma$  is a Lie subgroup of  $G$  as it is closed in  $G$ , and we will denote its Lie algebra by  $\mathfrak{g}_\gamma$ .

Then, for  $\varphi: G \times M \rightarrow M$  a left action, we have

**Lemma 2.18**

If  $\gamma: D \rightarrow M$  is analytic, then  $G_\gamma = G_{\gamma|_{D'}}$  holds for each interval  $D' \subseteq D$ .

PROOF: Since  $\gamma$  and  $g \cdot \gamma$  are analytic, we have

$$g \in G_{\gamma|_{D'}} \xrightarrow{\text{def}} (g \cdot \gamma)|_{D'} = \gamma|_{D'} \xrightarrow{\text{Lemma 2.1}} g \cdot \gamma = \gamma. \quad \blacksquare$$

From this, we immediately obtain

**Corollary 2.19**

If  $\gamma: D \rightarrow M$  is an analytic immersion, then

$$g \cdot \gamma \sim_\circ \gamma \implies g^{-1} \cdot q \cdot g \in G_\gamma \quad \forall q \in G_\gamma. \quad (7)$$

PROOF: Let  $q \in G_\gamma$ , and  $J, J' \subseteq I$  be open intervals with  $g \cdot \gamma(J) = \gamma(J')$ . Then, we have

$$q \cdot (g \cdot \gamma(t)) = g \cdot \gamma(t) \quad \forall t \in J \implies (g^{-1} \cdot q \cdot g) \cdot \gamma|_J = \gamma|_J \xrightarrow{\text{Lemma 2.18}} g^{-1} \cdot q \cdot g \in G_\gamma. \quad \blacksquare$$

Finally, let us show that

**Lemma 2.20**

Let  $\varphi$  be sated, and  $\gamma: K \rightarrow M$  an analytic embedding. If  $L = [\tau, l] \subseteq K = [\tau, k]$  holds, then

$$g \cdot \gamma(L) \subseteq \gamma(K) \quad \text{for} \quad g \in G_{\gamma(\tau)} \implies g \in G_\gamma.$$

PROOF: Since  $\gamma$  is an embedding, and  $\gamma(\tau)$  is fixed by  $g$ , we have

$$g \cdot \gamma(L) = \gamma(L') \quad \text{for} \quad L' = [\tau, l'] \subseteq K;$$

whereby, replacing  $g$  by  $g^{-1}$  if necessary, we can assume that  $L' \subseteq L$  holds. Then,  $\rho: L \rightarrow L', t \mapsto \gamma^{-1} \circ (g \cdot \gamma)(t)$  is positive as it fixes  $\tau$ , hence strictly increasing. Thus,

- ▷ If there is some  $t \in (\tau, l]$  with  $\rho(t) < t$ , then for  $\rho(t) < s < t$ , we have  $\rho(s) < \rho(t) < s < t$ .
- ▷ Thus, applying  $\rho$  successively, we obtain decreasing sequences  $\{s_n\}_{n \in \mathbb{N}}, \{t_n\}_{n \in \mathbb{N}} \subseteq (\tau, l]$  with  $t_{n+1} < s_n < t_n$  for all  $n \in \mathbb{N}$ .
- ▷ Then, if  $v \in [\tau, l]$  denotes their common limit,

$$\lim_n g^n \cdot \gamma(t) = \gamma(\rho^n(t)) = \lim_n \gamma(t_n) = \gamma(v) = \lim_n \gamma(s_n) = \gamma(\rho^n(s)) = \lim_n g^n \cdot \gamma(s)$$

contradicts that  $\gamma(v)$  is sated.

Thus,  $\rho(t) \geq t$  holds for all  $t \in L$ , hence  $L = L'$ , because  $\rho(l) = l' \leq l$  holds by assumption. Then,

- ▷ If there is some  $t \in [\tau, l)$  with  $t < \rho(t)$ , then for  $t < s < \rho(t)$ , we have  $t < s < \rho(t) < \rho(s)$ , whereby  $\rho(t)$  and  $\rho(s)$  are both contained in  $L$ , just because we have already shown that  $L = L' = \text{im}[\rho]$  holds.
- ▷ Then, applying  $\rho$  successively, we obtain increasing sequences  $\{s_n\}_{n \in \mathbb{N}}, \{t_n\}_{n \in \mathbb{N}} \subseteq [\tau, l)$  with  $t_n < s_n < t_{n+1}$  for all  $n \in \mathbb{N}$ , so that we can argue as above, in order to derive a contradiction to satedness of  $\varphi$ .

Consequently,  $\rho = \text{id}_L$ , hence  $g \in G_{\gamma|_L} = G_\gamma$  holds by Lemma 2.18. ■

## 2.5 Lie algebra generated curves

We close this section with some important facts concerning the maps (3).

### 2.5.1 Standard facts

Let us start with, cf. Lemma 5.6.2 in [4].

#### Lemma 2.21

If  $\gamma = \gamma_{\vec{g}}^x$  holds for some  $x \in M$  and  $\vec{g} \in \mathfrak{g} \setminus \mathfrak{g}_x$ , then

- $\gamma$  is analytic immersive.
- If  $\gamma$  is not injective, it is cyclic in the sense that there is  $\pi_{\vec{g}} \in \mathbb{R}_{>0}$  uniquely determined, such that

$$\gamma(t) = \gamma(t') \iff t = t' + n \cdot \pi_{\vec{g}} \text{ for some } n \in \mathbb{Z}.$$

PROOF: First observe that  $\gamma$  is immersive, because<sup>16</sup>

$$\dot{\gamma}(t) = d_e \varphi_{\exp(t \cdot \vec{g}) \cdot x}(\vec{g}) = 0 \iff \vec{g} \in \mathfrak{g}_{\exp(t \cdot \vec{g}) \cdot x} = \text{Ad}_{\exp(t \cdot \vec{g})}(\mathfrak{g}_x).$$

In fact, then  $\dot{\gamma}(t) = 0$  implies  $\vec{g} = \text{Ad}_{\exp(t \cdot \vec{g})}(\vec{c})$  for some  $\vec{c} \in \mathfrak{g}_x$ , hence  $\vec{c} = \text{Ad}_{\exp(-t \cdot \vec{g})}(\vec{g}) = \vec{g}$ , which contradicts the choice of  $\vec{g}$ . Now, assume that  $\gamma$  is not injective, and observe that

$$\gamma(t') = \gamma(t) \iff \gamma(t' + a) = \gamma(t + a) \quad \forall a \in \mathbb{R} \quad (8)$$

holds. Then, for  $\pi_{\vec{g}}$  the infimum of  $\{t \in \mathbb{R}_{>0} \mid \gamma(0) = \gamma(t)\} \subseteq \mathbb{R}$ , we have

▷  $\pi_{\vec{g}} > 0$  since  $\gamma$  is locally injective, as it is an immersion.

▷  $\gamma(\pi_{\vec{g}}) = \gamma(0)$  by continuity of  $\gamma$ , hence  $\gamma(n \cdot \pi_{\vec{g}}) = \gamma(0)$  for each  $n \in \mathbb{Z}$ , which follows inductively from (8).

Then, if  $\gamma(t) = \gamma(t')$  holds for some  $t, t' \in \mathbb{R}$ , we have  $\gamma(t - t') = \gamma(0)$  by (8), hence  $0 \leq (t - t') + n \cdot \pi_{\vec{g}} \leq \pi_{\vec{g}}$  for some  $n \in \mathbb{Z}$ . Thus, we have

$$\gamma(t - t') = \gamma(0) \xrightarrow{(8)} \gamma((t - t') + n \cdot \pi_{\vec{g}}) = \gamma(n \cdot \pi_{\vec{g}}) = \gamma(0),$$

so that minimality of  $\pi_{\vec{g}}$  shows that  $t - t' + n \cdot \pi_{\vec{g}} \in \{0, \pi_{\vec{g}}\}$ , hence  $t - t' = n' \cdot \pi_{\vec{g}}$  holds for some  $n' \in \mathbb{Z}$ . ■

Thus, for  $x \in M$  and  $\vec{g} \in \mathfrak{g} \setminus \mathfrak{g}_x$  fixed, we define the **period**  $\pi_{\vec{g}}$  of  $\vec{g}$

- by  $\pi_{\vec{g}} := \infty$  iff  $\gamma_{\vec{g}}^x$  is injective,
- as in the second part of Lemma 2.21 iff  $\gamma_{\vec{g}}^x$  is not injective.

Then, cf. Lemma 5.6.5 in [4]

#### Lemma 2.22

If  $\gamma: D \rightarrow M$  is non-constant analytic with  $\gamma(t)$  sated for each  $t \in D$ , then we have

$$\gamma|_{\text{dom}[\tau]} = \gamma_{\vec{g}}^x \circ \tau \implies \gamma = \gamma_{\vec{g}}^x \circ \bar{\tau}|_D,$$

if  $\tau: \text{dom}[\tau] \rightarrow \text{im}[\tau] \subseteq \mathbb{R}$  is an analytic map with  $\text{dom}[\tau]$  an interval, and  $\bar{\tau}$  its maximal analytic extension.

PROOF: Since  $\gamma$  is not constant, we must have  $\vec{g} \notin \mathfrak{g}_x$ . Then, for  $\bar{\gamma}: \bar{I} \rightarrow M$  the maximal analytic extension of  $\gamma$ , and  $\bar{\tau}: I \rightarrow I' \subseteq \mathbb{R}$ , we have

$$\bar{\gamma}|_{\text{dom}[\tau]} = \gamma_{\vec{g}}^x \circ \bar{\tau}|_{\text{dom}[\tau]} \xrightarrow{\text{Lemma 2.1}} \bar{\gamma}|_I = \gamma_{\vec{g}}^x \circ \bar{\tau}$$

by maximality of  $\bar{\gamma}$ . Thus, the claim is clear if  $D \subseteq I$  holds. In the other case, we find some  $t \in D \setminus I \subseteq \bar{I}$ , and a strongly monotonously increasing or decreasing sequence  $\{t_n\}_{n \in \mathbb{N}} \subseteq I$ , with  $\lim_n t_n = t \in D$ ; and conclude that

<sup>16</sup>Observe that  $\text{Ad}_g: \mathfrak{g}_x \rightarrow \mathfrak{g}_y$  holds for  $y = g \cdot x$ , because if  $\vec{g} = \text{Ad}_g(\vec{c})$  for  $\vec{c} \in \mathfrak{g}_x$ , we have  $\exp(t \cdot \vec{g}) \cdot y = g \cdot \exp(t \cdot \vec{c}) \cdot x = y$  for all  $t \in \mathbb{R}$ , hence  $\vec{g} \in \mathfrak{g}_y$ .

▷ If  $\lim_n \bar{\tau}(t_n) = t' \in \mathbb{R}$  exists, we have

$$\bar{\gamma}(t) = \lim_n \bar{\gamma}(t_n) = \lim_n \gamma_{\bar{g}}^x(\bar{\tau}(t_n)) = \gamma_{\bar{g}}^x(t').$$

Then, Lemma 2.2 shows that  $\bar{\gamma}|_J = \gamma_{\bar{g}}^x \circ \rho$  holds for an analytic map  $\rho: J \rightarrow D'$  with  $\rho(t) = t'$ , whereby we can assume that  $D'$  is contained in some open interval  $J'$  on which  $\gamma_{\bar{g}}^x$  is injective. Since  $\bar{\tau}$  is monotonous, we find an open interval  $J''$  containing  $t$ , such that  $\bar{\tau}(I \cap J'') \subseteq J'$  holds. Then, injectivity of  $\gamma_{\bar{g}}^x|_{J'}$  shows that  $\rho$  must coincide on  $I \cap J''$  with  $\bar{\tau}$ . Thus,  $\bar{\tau}$  extends to an open interval containing  $t$ , which contradicts its maximality.

▷ If  $\lim_n \bar{\tau}(t_n) = \pm\infty$  holds, we fix  $0 < d < \pi_{\bar{g}}$ , and modify  $\{t_n\}_{n \in \mathbb{N}}$  in such a way that

$$\bar{\tau}(t_n) = \bar{\tau}(t_0) \pm n \cdot d \quad \forall n \in \mathbb{N}$$

holds. Then, for  $z := \gamma(t_0)$  and  $g := \exp(d \cdot \bar{g})$ , we have

$$\lim_n g^n \cdot (g \cdot z) = \lim_n g^n \cdot z = \lim_n \gamma_{\bar{g}}^x(\bar{\tau}(t_n)) = \lim_n \gamma(t_n) = \gamma(t).$$

This contradicts that  $\gamma(t)$  is sated, because  $\gamma(t) \neq g \cdot z \neq z \neq \gamma(t)$  holds. In fact, we have  $g \cdot z \neq z$  by the choice of  $g$ , and since obviously  $g \cdot \gamma(t) = \gamma(t) = g^{-1} \cdot \gamma(t)$  holds, we also must have  $z \neq \gamma(t) \neq g \cdot z$ . ■

### Remark 2.23

To get an idea what might happen if  $\varphi$  is not sated, let  $G = \mathbb{R}_{>0}$ ,  $M = \mathbb{R}^n$ , and  $\varphi: (\lambda, x) \mapsto \lambda \cdot x$  be the multiplicative action. Then,  $x \in M$  is sated iff  $x \neq 0$  holds, and the exponential map of  $G$  is given by  $\lambda \mapsto e^\lambda$  for  $\lambda \in \mathbb{R} \cong \mathfrak{g}$ . Now,

▷ For  $x \neq 0$  and  $\lambda > 0$ , we have  $\gamma_\lambda^x(t) = e^{t \cdot \lambda} \cdot x$ , hence  $\text{im}[\gamma_\lambda^x] = \{s \cdot x \mid s > 0\}$  as  $\lim_{t \rightarrow -\infty} e^{t \cdot \lambda} = 0$  holds.

▷ Thus,  $\gamma = \gamma_\lambda^x \circ \rho$  cannot hold for  $\gamma: \mathbb{R} \rightarrow M$ ,  $t \mapsto t \cdot x$  and some analytic diffeomorphism  $\rho: \mathbb{R} \rightarrow (0, \infty)$ , but we have  $\gamma|_{(0, \infty)} = \gamma_\lambda^x \circ (1/\lambda \cdot \ln)$ .

Here, the reason why  $\text{im}[\gamma]$  is not contained in  $\text{im}[\gamma_\lambda^x]$ , is that  $\lim_{t \rightarrow -\infty} e^{t \cdot \lambda} \cdot x = 0 \in M$  exists; and, as the last point in the proof of Lemma 2.22 shows, this can only happen, because  $\varphi$  is not sated at the origin. ‡

### 2.5.2 Uniqueness

Finally, we want to clarify w.r.t. which  $x \in M$  and  $\bar{g} \in \mathfrak{g}$ , an analytic curve  $\gamma: D \rightarrow M$  can be Lie.

This is trivial if  $\gamma$  is constant, because then  $\gamma = \gamma_{\bar{g}}^x \circ \rho$  holds for any analytic map  $\rho: D \rightarrow D'$ , and each  $\bar{g} \in \mathfrak{g}_x$  for  $\{x\} = \text{im}[\gamma]$ ; alternatively, one can also choose  $y \in M$  and  $\bar{q} \in \mathfrak{g}$  arbitrary with  $\gamma_{\bar{q}}^y(t) = x$  for some  $t \in \mathbb{R}$ , and define  $\rho$  to be constant  $t$  on  $D$ .

Anyhow, if  $\gamma$  is non-constant, it is immersive on some open interval  $I \subseteq D$ , just because by analyticity of  $\dot{\gamma}$ , the set  $Z = \{t \in D \mid \dot{\gamma}(t) = 0\}$  must consist of isolated points in this case. Thus, if  $\gamma$  is Lie w.r.t.  $\gamma_{\bar{g}}^x$  and  $\gamma_{\bar{q}}^y$ , we have

$$\gamma_{\bar{g}}^x \circ \rho|_I = \gamma|_I = \gamma_{\bar{q}}^y \circ \rho'|_I \quad \implies \quad \gamma_{\bar{g}}^x \sim_\circ \gamma_{\bar{q}}^y,$$

as  $\rho|_I$  and  $\rho'|_I$  are necessarily immersive. Moreover,  $\gamma_{\bar{g}}^x$  and  $\gamma_{\bar{q}}^y$  are obviously non-constant, and for  $\varphi$  sated, we conclude that

### Lemma 2.24

Let  $\varphi$  be sated and  $\gamma_{\bar{g}}^x, \gamma_{\bar{q}}^y$  non-constant. Then,  $\gamma_{\bar{g}}^x \sim_\circ \gamma_{\bar{q}}^y$  implies  $\bar{q} \in \lambda \cdot \bar{g} + \mathfrak{g}_\gamma$  for some  $\lambda \neq 0$ .

PROOF: Define  $\gamma := \gamma_{\bar{g}}^x$  and  $\delta := \gamma_{\bar{q}}^y$ . Then,  $\gamma = \delta \circ \rho$  holds for some analytic diffeomorphism  $\rho: \mathbb{R} \rightarrow \mathbb{R}$  by Lemma 2.22; and replacing  $\bar{q}$  by  $-\bar{q}$  and  $\rho$  by  $-\rho$  if necessary, we can assume that  $\dot{\rho} > 0$  holds. Moreover, replacing  $y$  by  $\exp(\rho(0) \cdot \bar{q}) \cdot y$  and  $\rho$  by  $\rho - \rho(0)$ , we additionally can achieve that  $\rho(0) = 0$ . Then, we have

$$g_t := \exp(-\rho(t) \cdot \bar{q}) \cdot \exp(t \cdot \bar{g}) \in G_{\gamma(0)} \quad \forall t \in \mathbb{R},$$



because for each  $t \in \mathbb{R}$ , and each  $s \geq 0$ , we have

$$\begin{aligned} g_t \cdot \gamma(s) &= \exp(-\rho(t) \cdot \vec{q}) \cdot \gamma(t+s) = \exp(-\rho(t) \cdot \vec{q}) \cdot \delta(\rho(t+s)) \\ &= \exp(-\rho(t) \cdot \vec{q}) \cdot \delta(\rho(t) + \Delta(s)) = \delta(\Delta(s)) = \gamma(\rho^{-1}(\Delta(s))) = \gamma(\Delta'(s)) \end{aligned}$$

for homeomorphisms  $\Delta, \Delta': \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  with  $\Delta(0), \Delta'(0) = 0$ . Thus, for  $s > 0$ , we have  $g_t \cdot \gamma([0, s]) = \gamma([0, s'])$  for some  $s' > 0$ ; and for  $s$  suitably small, even  $s, s' < \pi_{\vec{g}}$  holds. Then,

- If  $s' \geq s$ , we apply Lemma 2.20 to  $g = g_t$ ,  $\tau = 0$ ,  $k = s'$ , and  $l = s$ , in order to conclude that  $g_t \in G_\gamma$  holds.
- If  $s' < s$ , we we apply Lemma 2.20 to  $g = g_t^{-1}$ ,  $\tau = 0$ ,  $k = s$ , and  $l = s'$ , in order to conclude that  $g_t^{-1} \in G_\gamma$ , hence  $g_t \in G_\gamma$  holds.

Then, the claim just follows by taking the derivative of  $g_t$  at  $t = 0$ . ■

Conversely,

**Lemma 2.25**

For each  $\lambda \neq 0$ ,  $\vec{c} \in \mathfrak{g}_\gamma$ ,  $s \in \mathbb{R}$ , we have  $\gamma_{\lambda \cdot \vec{g} + \vec{c}}^y = \gamma_{\vec{g}}^x \circ \rho$  for  $\rho: t \mapsto s + \lambda \cdot t$  and  $y := \gamma_{\vec{g}}^x(s)$ .

PROOF: The claim is clear if  $\gamma := \gamma_{\vec{g}}^x$  is constant, as then  $y = x$  and  $\lambda \cdot \vec{g} + \vec{c} \in \mathfrak{g}_x$  holds. In the other case, we let  $H$  denote the closure in  $G$ , of the group generated by the set  $O_\gamma := \{g \in G \mid g \cdot \gamma \sim_\circ \gamma\}$ .

Then,  $G_\gamma \subseteq H$  is a normal subgroup, because  $g^{-1} \cdot q \cdot g \in G_\gamma$  holds for each  $q \in G_\gamma$ , and each  $g \in O_\gamma$ , by Corollary 2.19. Thus,  $Q := H/G_\gamma$  is a Lie group, and the canonical projection map  $\pi: H \rightarrow Q$  is a Lie group homomorphism with  $\ker[d_e \pi] = \mathfrak{g}_\gamma$ . Now,  $\vec{g}$  is contained in the Lie algebra of  $H$ , because

$$\exp(s \cdot \vec{g}) \cdot \gamma(t) = \gamma(s+t) \quad \forall s, t \in \mathbb{R} \quad \implies \quad \exp(s \cdot \vec{g}) \in O_\gamma \subseteq H \quad \forall s \in \mathbb{R}.$$

Thus, for each  $t \in \mathbb{R}$ , and each  $\vec{c} \in \mathfrak{g}_\gamma$ , we have

$$\pi(\exp(t \cdot [\vec{g} + \vec{c}])) = \exp_q(t \cdot d_e \pi(\vec{g})) = \pi(\exp(t \cdot \vec{g})) \quad \implies \quad \exp(t \cdot [\vec{g} + \vec{c}]) = \exp(t \cdot \vec{g}) \cdot h_{t, \vec{c}} \quad (9)$$

for some  $h_{t, \vec{c}} \in G_\gamma$ , hence

$$\gamma_{\lambda \cdot \vec{g} + \vec{c}}^y(t) = \exp(t \cdot \lambda \cdot [\vec{g} + \frac{1}{\lambda} \cdot \vec{c}]) \cdot y \stackrel{(9)}{=} \exp(t \cdot \lambda \cdot \vec{g}) \cdot y = \gamma(s + \lambda \cdot t) \quad \forall t \in \mathbb{R}. \quad \blacksquare$$

Then, combining the previous two lemmas, we obtain

**Corollary 2.26**

Let  $\varphi$  be sated and  $\gamma_{\vec{g}}^x, \gamma_{\vec{q}}^y$  non-constant. If  $\gamma_{\vec{q}}^y = \gamma_{\vec{g}}^x \circ \rho$  holds for some analytic diffeomorphism  $\rho: \mathbb{R} \rightarrow \mathbb{R}$ , we necessarily have

$$\rho: t \mapsto s + \lambda \cdot t, \quad \vec{q} \in \lambda \cdot \vec{g} + \mathfrak{g}_\gamma, \quad y = \gamma_{\vec{g}}^x(s) \quad \text{for} \quad \lambda = \dot{\rho}(0) \quad \text{and} \quad s = \rho(0). \quad (10)$$

In particular,  $\gamma_{\vec{q}}^y = \gamma_{\vec{g}}^x \circ \mu$  holds exactly for the analytic diffeomorphisms  $\mu := \Delta + \rho$ , for  $\Delta \in \mathbb{Z} \cdot \pi_{\vec{g}}$ .

PROOF: We have  $\vec{q} = \lambda \cdot \vec{g} + \vec{c}$  for some  $\lambda \neq 0$ , and some  $\vec{c} \in \mathfrak{g}_\gamma$  by Lemma 2.24. Then, for  $\rho'$  the respective analytic diffeomorphism from (10), by Lemma 2.25, we have

$$\gamma_{\vec{q}}^y \rightsquigarrow_{0, s} \gamma_{\vec{g}}^x \quad \text{w.r.t.} \quad \rho' \quad \text{and} \quad \rho \quad \implies \quad \rho' = \rho,$$

as we have clarified in Subsection 2.3. ■

In particular, combining Lemma 2.25 with Corollary 2.26, we see that in the non-constant case:

- $\gamma_{\vec{q}}^x = \gamma_{\vec{g}}^x$  holds iff we have  $\vec{q} \in \vec{g} + \mathfrak{g}_\gamma$  for  $\gamma := \gamma_{\vec{g}}^x$ . In particular,  $\pi_{\vec{g} + \vec{c}} = \pi_{\vec{g}}$  holds for each  $\vec{c} \in \mathfrak{g}_\gamma$ .
- $\gamma_{\vec{q}}^x = \gamma_{\vec{g}}^x \circ \rho$  holds for an analytic diffeomorphism  $\rho: \mathbb{R} \rightarrow \mathbb{R}$  with  $\rho(0) = 0$  iff we have  $\rho: t \mapsto \Delta + \lambda \cdot t$ , and  $\vec{q} \in \lambda \cdot \vec{g} + \mathfrak{g}_\gamma$ , for some  $\lambda \neq 0$ , and some  $\Delta \in \mathbb{Z} \cdot \pi_{\vec{g}}$ .

In addition to that, we have shown

**Corollary 2.27**

Let  $\varphi$  be sated and  $\gamma$  non-constant analytic. Then, if  $\gamma$  is Lie w.r.t. some  $x \in M$  and  $\vec{g} \in \mathfrak{g}$ , it is Lie w.r.t. some  $y \in M$  and  $\vec{q} \in \mathfrak{g}$  iff  $y \in \exp(\text{span}_{\mathbb{R}}(\vec{g})) \cdot x$  and  $\vec{q} \in \lambda \cdot \vec{g} + \mathfrak{g}_\gamma$  holds for some  $\lambda \neq 0$ .

### 3 The Classification

In this section, we will prove the classification Theorem 3.6, stating that an analytic curve is either free or Lie, provided that  $\varphi$  is regular. Then, in Section 4, free curves will be shown to be discretely generated by the symmetry group, cf. Section 1. This section is organized as follows:

- In the first part, we will show that an analytic immersive curve is (up to parametrization) locally of the form (3) if it fulfils a special approximation property.
- In the second part, we will introduce the notion of a free curve, and show that each analytic immersive curve which is not of this type, has a local self-similarity property.
- In the last part, this self-similarity property, will be shown to be equivalent to the approximation property introduced in the first part, finally providing us with our classification Theorem 3.6.

So, for the rest of this section, let  $\varphi: G \times M \rightarrow M$  denote some fixed left action.

#### 3.1 Lie curves

In Lemma 2.21, we have seen that the maps (3) are analytic immersions for  $\vec{g} \notin \mathfrak{g}_x$ . We will now show that

##### Proposition 3.1

*If  $\varphi$  is sated, an analytic immersion  $\gamma: I \rightarrow M$  is Lie if it is Lie at some  $\tau \in I$ .*

Here,

##### Definition 3.2

An analytic immersion  $\gamma: I \rightarrow M$  is said to be Lie at  $\tau \in I$  iff, for  $x := \gamma(\tau)$ , there exists a **faithful** sequence  $G \setminus G_x \supseteq \{g_n\}_{n \in \mathbb{N}} \rightarrow e$ , such that  $\tau$  is an accumulation point of  $T := \{\tau < t \in I \mid x \rightarrow \gamma(t)\}$ . Here,

- Faithful means that we find  $\{\vec{g}_n\}_{n \in \mathbb{N}} \subseteq \mathfrak{g} \setminus \mathfrak{g}_x$  and  $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_{>0}$ , such that

$$g_n = \exp(\lambda_n \cdot \vec{g}_n) \quad \forall n \in \mathbb{N} \quad \text{as well as} \quad \lim_n \lambda_n = 0 \quad \text{and} \quad \lim_n \vec{g}_n = \vec{g} \in \mathfrak{g} \setminus \mathfrak{g}_x \quad \text{holds.}$$

- We write  $x \rightarrow \gamma(t)$  for  $\tau < t \in I$  iff for each  $n_0 \in \mathbb{N}$ , and for each  $\epsilon > 0$  with  $\tau < t - \epsilon$ , we find  $n \geq n_0$  as well as  $m \in \mathbb{N}$  with<sup>17</sup>

$$(g_n)^k \cdot x \in \gamma((\tau, t]) \quad \forall k = 1, \dots, m \quad \text{and} \quad (g_n)^m \cdot x \in \gamma((t - \epsilon, t]). \quad (11)$$

Obviously, the above definition is local in the sense that  $\gamma$  is Lie at  $\tau$  iff  $\gamma|_J$  is Lie at  $\tau$  for each open interval  $J \subseteq I$  containing  $\tau$ . Then, Proposition 3.1 is an immediate consequence of Lemma 2.22 and

##### Lemma 3.3

*If  $\gamma: I \rightarrow M$  is Lie at  $\tau \in I$ , then  $\gamma|_J$  is Lie for some open interval  $J \subseteq I$  containing  $\tau$ .*

PROOF: By locality, we can assume that  $\gamma$  is equicontinuous and an embedding, and that  $\text{im}[\gamma]$  is contained in some chart  $(O, \psi)$  around  $x := \gamma(\tau)$ . Moreover, since  $\varphi_x \circ \exp$  is continuous and  $\lim_n \vec{g}_n = \vec{g}$  holds, we find  $0 < l < \pi_{\vec{g}}$  and  $n_0 \in \mathbb{N}$ , such that the images of

$$\delta := \gamma_{\vec{g}}^x|_L \quad \text{and} \quad \delta_n := \gamma_{\vec{g}_n}^x|_L \quad \forall n \geq n_0 \quad \text{with} \quad L := [0, l]$$

are contained in  $O$ . Thus, we can assume that  $M = \psi(O)$  holds, and that its topology is determined by the euclidean norm  $\|\cdot\|$  on  $\mathbb{R}^{\dim(M)}$ . Then,  $\{\delta_n\}_{n \geq n_0} \rightarrow \delta$  converges uniformly on  $L$ .

Now, assume that the statement is wrong:

- ▷ Since, by Lemma 2.21,  $\delta$  is an analytic embedding,  $x$  cannot be an accumulation point of  $\text{im}[\gamma] \cap \text{im}[\delta]$  by Lemma 2.3 and Lemma 2.4.

---

<sup>17</sup>In simple words,  $\gamma(t)$  can be “arbitrarily well” approximated by successive shifts of  $x$  through  $\gamma((\tau, t])$  by some “arbitrarily small”  $g_n$ .

- ▷ Thus,  $\text{im}[\gamma] \cap \delta(K) = \emptyset$  must hold for some compact interval  $K \subseteq (0, l]$ , hence  $d := \text{dist}(\delta(K), \text{im}[\gamma]) > 0$ .
- ▷ Then, since  $\{\delta_n\}_{n \geq n_0} \rightarrow \delta$  converges uniformly, increasing  $n_0$  if necessary, we can assume that  $\|\delta_n - \delta\|_\infty < d$ , hence  $\delta_n(K) \cap \text{im}[\gamma] = \emptyset$ , holds for each  $n \geq n_0$ .
- ▷ Moreover, since  $\lim_n \lambda_n = 0$  holds, increasing  $n_0$  once more if necessary, for each  $n \geq n_0$ , we find  $m(n) \in \mathbb{N}_{>0}$  with  $m(n) \cdot \lambda_n \in K$ , hence  $\delta_n(m(n) \cdot \lambda_n) \notin \text{im}[\gamma]$ .

Then, for  $\Delta > 0$ , choose  $n'_0 > n_0$  with  $\|\delta|_L - \delta_n|_L\|_\infty < \Delta$  for all  $n \geq n'_0$ . Moreover, let  $\epsilon > 0$  be such that  $|t - t'| < \epsilon$  implies  $\|\gamma(t) - \gamma(t')\| < \Delta$ . Then, for  $t \in T$  fixed, we find  $m > 0$  and  $n \geq n'_0$ , such that (11) holds w.r.t.  $\epsilon$ , hence

$$\|\gamma(t) - \delta_n(m \cdot \lambda_n)\| = \|\gamma(t) - \exp(\lambda_n \cdot \vec{g}_n)^m \cdot x\| = \|\gamma(t) - (g_n)^m \cdot x\| \stackrel{(11)}{<} \Delta.$$

For this observe that  $m \cdot \lambda_n \in L$  holds, because  $m(n) \cdot \lambda_n \in K$  implies  $m < m(n)$ . In fact, by (11),  $m(n) \leq m$  implies  $\delta_n(m(n) \cdot \lambda_n) = g^{m(n)} \cdot x \in \text{im}[\gamma]$ , which contradicts the definition of  $m(n)$ .

Thus, we have

$$\|\gamma(t) - \delta(m \cdot \lambda_n)\| \leq \|\gamma(t) - \delta_n(m \cdot \lambda_n)\| + \|\delta_n(m \cdot \lambda_n) - \delta(m \cdot \lambda_n)\| < 2\Delta,$$

and since this holds for each  $t \in T$ , and each  $\Delta > 0$ , we have  $\gamma(T) \subseteq \delta(L)$ . This, however, contradicts the assumption that  $x$  is not an accumulation point of  $\text{im}[\gamma] \cap \text{im}[\delta]$ . ■

### 3.2 Free curves

We will start our considerations with the definition of a free curve. Then, we will show that an analytic immersive curve which is not free, has some special local self-similarity property if  $\varphi$  is sated. In Subsection 3.3, this property then will be shown to be equivalent to Definition 3.2 if  $\varphi$  is even regular, which will finally provide us with

#### Proposition 3.4

*If  $\varphi$  is regular, an analytic immersive curve is either free or Lie.*

Here, by a free curve, we understand the following:

#### Definition 3.5 (Free curve)

- A free segment is an analytic immersion  $\gamma: D \rightarrow M$  with

$$g \cdot \gamma \sim_o \gamma \quad \text{for } g \in G \quad \implies \quad g \cdot \gamma = \gamma \quad \iff \quad g \in G_\gamma. \quad (12)$$

Obviously, each restriction of a free segment to some interval is a free segment as well.

- A **free** curve is an analytic curve  $\gamma: D \rightarrow M$  with  $\gamma|_{D'}$  a free segment for some interval  $D' \subseteq D$ ; in particular, then  $\gamma$  is not constant.

Then, from Proposition 3.4, we easily obtain our classification

#### Theorem 3.6

*If  $\varphi$  is regular, an analytic curve is either free or Lie, whereby the uniqueness statement from Corollary 2.27 holds in the second case.*

PROOF: Each constant curve is Lie, but not free, because it cannot admit any analytic immersive subcurve.

Thus, let  $\gamma: D \rightarrow M$  be non-constant and analytic, and define  $Z = \{t \in D \mid \dot{\gamma}(t) = 0\}$ . Then,  $Z$  consist of isolated points (and has no limit point in  $D$ ), just by analyticity of  $\dot{\gamma}$ . Consequently,  $\gamma|_J$  is analytic immersive for some open interval  $J \subseteq D$ , hence either free or Lie by Proposition 3.4. Now,

- If  $\gamma|_J$  is Lie, then  $\gamma$  is Lie by Lemma 2.22. Thus, each analytic immersive subcurve of  $\gamma$  is Lie as well, so that  $\gamma$  cannot free by Proposition 3.4.
- If  $\gamma|_J$  is free, so is  $\gamma$ ; and then  $\gamma$  cannot be Lie, because otherwise each subcurve of  $\gamma$ , hence  $\gamma|_J$  is Lie. ■

In particular,

**Corollary 3.7**

*If  $\varphi$  is regular and  $\gamma$  is free, so is each analytic immersive subcurve of  $\gamma$ .*

PROOF: In fact, otherwise  $\gamma$  is Lie by Proposition 3.4 and Lemma 2.22, which contradicts Theorem 3.6.  $\blacksquare$

**Remark 3.8**

In Section 4, we will show that each free analytic immersion  $\gamma: I \rightarrow M$  admits a natural decomposition into countably many free segments, mutually (and uniquely) related by the group action. If  $\gamma$  is non-constant analytic, we can define  $Z$  as in the proof of Theorem 3.6, and conclude that

- If  $Z = \emptyset$ , then  $\gamma$  is analytic immersive.
- If  $Z \neq \emptyset$ , then, since  $Z$  consists of isolated points and admits no limit point in  $I$ , we have<sup>18</sup>  $Z = \{t_n\}_{n_- \leq n \leq n_+}$  for  $-\infty \leq n_- \leq 0 \leq n_+ \leq \infty$  with  $t_m < t_n$  if  $m < n$ , such that
  - if  $n_- = -\infty$  holds, then for each  $t \in I$ , we have  $t_n < t$  for some  $n_- \leq n \leq n_+$ ,
  - if  $n_+ = \infty$  holds, then for each  $t \in I$ , we have  $t < t_n$  for some  $n_- \leq n \leq n_+$ .

Then, the restriction of  $\gamma$  to the connected components of  $I \setminus Z$  is analytic immersive by definition, as well as free by Corollary 3.7. Thus, our decomposition results for analytic immersions apply to each of these subcurves; and it is then the task to figure out, in which way the respective decompositions glue together at the points  $t_n$ . Alternatively, we can also change the definition of  $\sim_\circ$  (cf. Subsection 2.2.2), as well as Definition 3.5 in that way that we replace “analytic immersive” by “non-constant analytic”, and then go through the arguments of Section 4. But, then we will have some difficulties with certain uniqueness statements proven there, as those rely on Lemma 2.11, which, in turn, relies on Corollary 2.9.

For this observe that a non-constant analytic curve can “inverse its direction” at the points  $t_n$ ,<sup>19</sup> which is impossible for the points in  $I \setminus Z$  by Corollary 2.9. This issue, however, should be of rather combinatorial nature, because if  $\gamma$  is immersive on  $(i', t_n)$  and  $(t_n, i)$ , and “inverses its direction” at  $t_n$ , then  $\gamma|_{(i', t_n)}$  is either a subcurve of  $\gamma|_{(t_n, i)}$  or vice versa. In any case, however, the crucial question one has to answer first, is whether the following statement holds or not:

*Let  $\gamma: I \rightarrow M$  and  $\gamma': I' \rightarrow M$  be non-constant analytic, and only non-immersive at  $t \in I$  and  $t' \in I'$ , respectively. Moreover, assume that  $\gamma = \gamma' \circ \rho$  holds for some positive analytic diffeomorphism  $\rho: I \cap (-\infty, t) \rightarrow I' \cap (-\infty, t')$ , and that both  $\gamma$  and  $\gamma'$  do not “inverse their direction” at  $t$  and  $t'$ , respectively. Then,  $\gamma = \gamma' \circ \tau$  holds for some positive analytic diffeomorphism  $\tau: (t, \epsilon) \rightarrow (t', \epsilon')$ .*  $\ddagger$

Now, let us start to collect the statements that we will need to prove Proposition 3.4. First of all, it is straightforward that

**Lemma 3.9**

*If an analytic immersion  $\gamma: D \rightarrow M$  is Lie, it is not free.*

PROOF: We have  $\gamma = \gamma_g^x \circ \rho$  for some analytic diffeomorphism  $\rho: D \rightarrow \rho(D) \subseteq \mathbb{R}$ , so that the stabilizers of  $\gamma$  and  $\gamma_g^x$  coincide. Moreover, if  $D' \subseteq D$  is some interval, then  $\gamma|_{D'}$  is an embedding for some compact interval  $K \subseteq D'$ , so that  $\gamma_g^x|_L$  is an embedding for  $L := \rho(K)$ . Then, if we write  $L = [r, r + 2\epsilon]$  for  $r \in \mathbb{R}$  and  $\epsilon > 0$ , the formula

$$\exp(s \cdot \vec{g}) \cdot \gamma_g^x(t) = \gamma_g^x(s + t) \quad \forall s, t \in \mathbb{R} \quad (13)$$

implies  $g := \exp(\epsilon \cdot \vec{g}) \notin G_\gamma$  by injectivity of  $\gamma_g^x|_L$ , as well as

$$g \cdot \gamma_g^x([r, r + \epsilon]) \stackrel{(13)}{=} \gamma_g^x([r + \epsilon, r + 2\epsilon]) \implies g \cdot \gamma_g^x|_L \sim_\circ \gamma_g^x|_L \implies g \cdot \gamma|_{D'} \sim_\circ \gamma|_{D'}. \quad \blacksquare$$

<sup>18</sup>If  $n_- = -\infty$ , then  $n_- \leq n$  means  $n \in \mathbb{Z}$ , and analogously for  $n_+$ .

<sup>19</sup>More precisely, by this we mean that  $\gamma|_{(t_n, i)} = \gamma \circ \rho$  holds for some negative analytic diffeomorphism  $\rho: (t_n, i) \rightarrow (i', t_n)$ . For instance, compose some analytic immersive curve  $(-\epsilon, r^2) \rightarrow M$  with the analytic map  $(-r, r) \ni t \mapsto t^2$ .

For the rest of this subsection, let  $\varphi$  be sated. Then,

**Proposition 3.10**

If an analytic immersion  $\gamma: I \rightarrow M$  is not free, it is continuously generated at each  $\tau \in I$ .

Here,

**Definition 3.11**

An analytic immersion  $\gamma: I \rightarrow M$  is said to be continuously generated at  $\tau \in I$  iff for each compact interval of the form  $[\tau, k] \subseteq I$ , we find some  $g \in G \setminus G_{\gamma(\tau)}$  with  $g \cdot \gamma|_{[\tau, k]} \sim_{\circ} \gamma|_{[\tau, k]}$ .

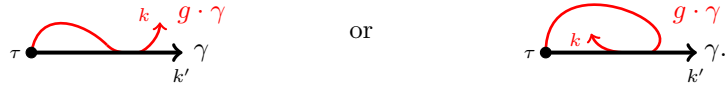
Now, before we come to the proof of Proposition 3.10, we first need to show

**Lemma 3.12**

Let  $\gamma: I \rightarrow M$  be an analytic embedding,  $\tau \in I$ , as well as  $K = [\tau, k] \subseteq I$  and  $K' = [\tau, k'] \subseteq I$ . Then,

$$g \cdot \gamma|_K \sim_{\circ} \gamma|_{K'} \quad \text{for } g \in G_{\gamma(\tau)} \setminus G_{\gamma} \quad \implies \quad g \cdot \gamma(k) \in \gamma(K').$$

PROOF: Let  $J \subseteq K$  and  $J' \subseteq K'$  be open intervals with  $g \cdot \gamma(J) = \gamma(J')$ . Then, depending on whether  $\rho := \gamma^{-1} \circ (g \cdot \gamma|_J)$  is positive or negative, we either we have



More precisely, let  $[c', c] = C := K \cap \bar{\rho}^{-1}(K')$  be as in Lemma 2.7. Then, we have

$$\dot{\rho} > 0 \quad \implies \quad c' = \tau = \bar{\rho}(c') \quad \implies \quad g \cdot \gamma([\tau, c]) \subseteq \gamma([\tau, k']) \quad \xrightarrow{\text{Lemma 2.20}} \quad g \in G_{\gamma},$$

which contradicts the choice of  $g$ , so that  $\dot{\rho} < 0$  must hold. For the first implication observe that

▷ If  $c' > \tau$ , we have  $\bar{\rho}(c') = \tau$  by (5) as  $\dot{\rho} > 0$  holds. This, however, contradicts injectivity of  $g \cdot \gamma$ , because

$$\bar{\rho}(c') = \tau \quad \text{and} \quad g \in G_{\gamma(\tau)} \quad \implies \quad g \cdot \gamma(c') = \gamma(\bar{\rho}(c')) = \gamma(\tau) = g \cdot \gamma(\tau).$$

▷ Thus, we have  $c' = \tau$ , and then injectivity of  $\gamma$  shows

$$\gamma(\tau) = g \cdot \gamma(\tau) = g \cdot \gamma(c') = \gamma(\bar{\rho}(c')) \quad \implies \quad \tau = \bar{\rho}(c').$$

Consequently, we have  $\dot{\rho} < 0$ , hence  $c = k$ , so that  $g \cdot \gamma(k) = \gamma(\bar{\rho}(c)) \in \gamma(K')$  holds. In fact, since  $\dot{\rho} < 0$ ,

$$c < k \quad \xrightarrow{(5)} \quad \bar{\rho}(c) = \tau \quad \implies \quad g \cdot \gamma(\tau) = \gamma(\tau) = \gamma(\bar{\rho}(c)) = g \cdot \gamma(c),$$

which contradicts that  $g \cdot \gamma$  is injective as  $\tau < c$  holds. ■

Then, since analytic immersions are locally embeddings, Proposition 3.10 is clear from the second part of

**Corollary 3.13**

Let  $\gamma: I \rightarrow M$  be an analytic embedding, and let  $\tau \in I$ .

1) If  $\{g_n\}_{n \in \mathbb{N}} \subseteq G \setminus G_{\gamma}$  is a sequence, and  $\{k_n\}_{n \in \mathbb{N}} \subseteq I \cap (\tau, \infty)$  decreasing with  $\lim_n k_n = \tau$ , then

$$g_n \cdot \gamma|_{[\tau, k_n]} \sim_{\circ} \gamma|_{[\tau, k_n]} \quad \forall n \in \mathbb{N} \quad (14)$$

implies that  $g_n \in G_{\gamma(\tau)}$  only holds for finitely many  $n \in \mathbb{N}$ .

2) If  $\gamma|_{[\tau, k]}$  is not a free segment for all  $\tau < k \leq l$  for some  $\tau < l \in I$ , then  $\gamma$  is continuously generated at  $\tau$ .

PROOF: 1) Let  $k_0 < k \in I$ , and assume that the statement is wrong. Then, passing to a subsequence, we can assume that  $\{g_n\}_{n \in \mathbb{N}} \subseteq G_{\gamma(\tau)} \setminus G_\gamma$  holds, and then (14) in combination with Lemma 3.12 (applied to each  $t \in [k_n, k]$  for  $K = [\tau, k]$  and  $K' = [\tau, k_n]$ ) shows that  $g_n \cdot \gamma([k_n, k]) \subseteq \gamma([\tau, k_n])$  holds for each  $n \in \mathbb{N}$ . Consequently, we have

$$g_n \cdot \gamma([k_0, k]) \subseteq g_n \cdot \gamma([k_n, k]) \subseteq \gamma([\tau, k_n]) \quad \forall n \in \mathbb{N},$$

which contradicts that  $\gamma(\tau)$  is sated.

- 2) Let  $\{k_n\}_{n \in \mathbb{N}} \subseteq (\tau, k] \subseteq I$  be decreasing with  $\lim_n k_n = \tau$ , and choose  $g_n \in G \setminus G_\gamma$  with  $g_n \cdot \gamma|_{[\tau, k_n]} \sim_\circ \gamma|_{[\tau, k_n]}$ , for each  $n \in \mathbb{N}$ . Then, Part 1) shows that  $\{g_n\}_{n \geq n_0} \subseteq G \setminus G_{\gamma(\tau)}$  holds for some  $n_0 \in \mathbb{N}$ , from which the claim is clear.  $\blacksquare$

### 3.3 The regular case

We now are going to prove Proposition 3.4, whereby we basically will have to show that, for regular actions, continuously generatedness implies Lieness, i.e., that

#### Proposition 3.14

Let  $\varphi$  be sated, and  $\gamma: I \rightarrow M$  continuously generated at  $\tau \in I$ . Then,  $\gamma$  is Lie if  $\gamma(\tau)$  is stable.

In fact, combining this with Proposition 3.10 and Lemma 3.9, we immediately obtain the

PROOF (OF PROPOSITION 3.4): If  $\gamma: D \rightarrow M$  is Lie, it is not free by Lemma 3.9. Conversely, if  $\gamma$  is not free, then  $\gamma' := \gamma|_I$  is not free for  $I \subseteq D$  some fixed open interval. Then,  $\gamma'$  is continuously generated at some (each)  $\tau \in I$  by Proposition 3.10, hence Lie by Proposition 3.14. Consequently,  $\gamma$  is Lie by Lemma 2.22.  $\blacksquare$

Now, before we come to the proof of Proposition 3.14, let us first discuss what might go wrong if the point  $\gamma(\tau)$  is not stable. For this,

#### Example 3.15

Let us consider the situation in Example 2.16.2 for  $\lambda$  irrational, with what  $\varphi$  is sated, but not stable at any point. Then,  $\gamma: \mathbb{R} \rightarrow \mathbb{T}^2$ ,  $t \mapsto (1, e^{2\pi t \cdot i})$  is continuously generated at 0, but not Lie.

- In fact,  $\gamma$  is not Lie, because it is not contained in the orbit of  $e \in \mathbb{T}^2$  under  $\varphi$ . For this, let  $\mu \in \mathbb{R}$  be such that  $1, \lambda, \mu$  are  $\mathbb{Q}$ -independent. Then,

$$\gamma(\mu) = \varphi(t, e) \quad \text{for some } t \in \mathbb{R} \quad \implies \quad (1, e^{2\pi \mu \cdot i}) = (e^{2\pi t \cdot i}, e^{2\pi t \lambda \cdot i}),$$

hence  $t \in \mathbb{Z}$  as  $1 = e^{2\pi t \cdot i}$ ; with what  $e^{2\pi \mu \cdot i} = e^{2\pi t \lambda \cdot i}$  implies that  $1, \lambda, \mu$  are  $\mathbb{Q}$ -dependent.

- Anyhow,  $\gamma$  is continuously generated at 0, because  $\{v^n\}_{n \in \mathbb{Z}} \subseteq \mathbb{T}^2$  is dense in  $U(1)$  for  $v := e^{2\pi \lambda \cdot i}$ , by Kronecker's theorem since  $1, \lambda$  are  $\mathbb{Q}$ -independent. In fact, then for  $0 < k < 1$  fixed, and  $\epsilon := k/4$ , we find  $n \in \mathbb{Z}$  with  $v^n = e^{2\pi s \cdot i}$  for some  $-\epsilon < s < \epsilon$ , hence

$$n \cdot \gamma(t) = \gamma(t + s) \quad \forall t \in \mathbb{R} \quad \implies \quad n \cdot \gamma([\epsilon, 2\epsilon]) = \gamma([s + \epsilon, s + 2\epsilon]).$$

Thus, we have  $n \cdot \gamma|_{[0, k]} \sim_\circ \gamma|_{[0, k]}$ , because  $[s + \epsilon, s + 2\epsilon] \subseteq [0, 3\epsilon]$  holds. For the formula on the left hand side, observe that  $n \cdot \gamma(t) = (1, v^n \cdot e^{2\pi t \cdot i})$  holds for all  $n \in \mathbb{Z}$ .  $\ddagger$

Now, for the rest of this section, let  $\varphi$  be sated. Moreover, let  $\gamma: I \rightarrow M$  denote some fixed analytic immersion which is continuously generated at  $\tau \in I$ , for  $x := \gamma(\tau)$  in addition stable. Then,

- a) In order to prove Proposition 3.14, by Proposition 3.1 and Lemma 2.22, it suffices to show that we find a bounded open interval  $J \subseteq I$  containing  $\tau$ , such that  $\gamma|_J$  is Lie at  $\tau$  or that  $\gamma \circ i|_J$  is Lie at  $i(\tau)$ , for  $i: J = (j', j) \rightarrow (j', j)$ ,  $t \mapsto j' + j - t$ .

In particular, in the following, we (can and) will assume that  $\gamma$  is an embedding, and that  $I$  is bounded.

Now, since  $\gamma$  is continuously generated at  $\tau \in I$ , we find and fix a sequence  $\{(g_n, K_n, J_n)\}_{n \in \mathbb{N}}$ , for

$$I \supset K := K_0 \supset J_0 \supset K_1 \supset J_1 \supset K_2 \supset J_2 \supset \dots$$

a shrinking collection of neighbourhoods of  $\tau$  with  $\bigcap_{n \in \mathbb{N}} K_n = \{\tau\}$ , as well as  $\{g_n\}_{n \in \mathbb{N}} \subseteq G \setminus G_{\gamma(\tau)}$ , such that we have

$$g_n \cdot \gamma|_{J_n} \sim_{\circ} \gamma|_{J_n} \quad \forall n \in \mathbb{N}. \quad (15)$$

Of course, here  $J_n$  is meant to be open, and  $K_n$  to be compact for each  $n \in \mathbb{N}$ .

We now are going to modify  $\{g_n\}_{n \in \mathbb{N}}$  in such a way that  $\gamma|_J$  is Lie at  $\tau$  or that  $\gamma \circ i|_J$  is Lie at  $i(\tau)$  w.r.t. this sequence, for  $J$  some open interval containing  $\tau$ . The next lemma then basically collects all the information that we will need.

**Lemma 3.16**

Let  $\{(g_n, K_n, J_n)\}_{n \in \mathbb{N}}$  be as above. Then, for each compact neighbourhood  $A \subseteq I$  of  $\tau$ , we find a compact neighbourhood  $B \subseteq I$  of  $\tau$ , such that  $g_n \cdot \gamma(B) \subseteq \gamma(A)$  holds for all  $n \geq m$ , for some  $m \in \mathbb{N}$ .

PROOF: Write  $A = [a', a]$ , as well as  $K_n = [k'_n, k_n]$  for each  $n \in \mathbb{N}$ . Then, if the statement is wrong, we find  $\iota: \mathbb{N} \rightarrow \mathbb{N}$  injective and increasing with  $g_{\iota(n)} \cdot \gamma(K_n) \not\subseteq \gamma(A)$  for each  $n \in \mathbb{N}$ . Then, (15) shows that

$$\gamma|_{K_{\iota(n)}} \sim_{\circ} g_{\iota(n)} \cdot \gamma|_{K_{\iota(n)}} \xrightarrow{n \leq \iota(n)} \gamma|_{K_n} \sim_{\circ} g_{\iota(n)} \cdot \gamma|_{K_n} \implies \gamma|_{J_n} = g_{\iota(n)} \cdot \gamma \circ \rho_n$$

for some analytic diffeomorphism  $\rho_n: K_n \supseteq J_n \rightarrow J'_n \subseteq K_n$ . Now, let  $m \in \mathbb{N}$  be such large that  $K_m \subseteq A$  holds, and fix some  $t_n \in J_n$  for each  $n \geq m$ . Then,

$$a' < k'_m \leq k'_n < t_n < k_n \leq k_m < a$$

holds for each  $n \geq m$ , hence

$$\begin{aligned} \gamma([a', t_n]) &\subseteq g_{\iota(n)} \cdot \gamma(K_n) &\implies & \gamma([a', k'_m]) \subseteq g_{\iota(n)} \cdot \gamma(K_n) &\quad \text{or} \\ \gamma([t_n, a]) &\subseteq g_{\iota(n)} \cdot \gamma(K_n) &\implies & \gamma([k_m, a]) \subseteq g_{\iota(n)} \cdot \gamma(K_n) \end{aligned}$$

by Corollary 2.8. Consequently,  $g_n^{-1} \cdot \gamma([a', k'_m]) \subseteq \gamma(K_n)$  or  $g_n^{-1} \cdot \gamma([k_m, a]) \subseteq \gamma(K_n)$  holds for infinitely many  $n \in \mathbb{N}$ , which contradicts that  $\gamma(\tau)$  is sated, as  $\bigcap_n K_n = \{\tau\}$  holds. ■

In particular, we have

**Corollary 3.17**

Let  $\{(g_n, K_n, J_n)\}_{n \in \mathbb{N}}$  be as above. Then,

- 1) For each compact neighbourhood  $L \subseteq I$  of  $\tau$ , we find  $n_0 \in \mathbb{N}$  with  $g_n \cdot \gamma(K_n) \subseteq \gamma(L)$  for each  $n \geq n_0$ .
- 2) We find some compact neighbourhood  $L \subseteq K$  of  $\tau$  and  $n_0 \in \mathbb{N}$ , with  $g_n \cdot \gamma(L) \subseteq \gamma(K)$  for each  $n \geq n_0$ .

PROOF: 1) In Lemma 3.16, let  $A = L$ , and  $n_0 \geq m$  be such large that  $K_n \subseteq B$  holds for all  $n \geq n_0$ .

- 2) In Lemma 3.16, let  $A = K$ , and define  $L := B \cap A$ , as well as  $n_0 := m$ . ■

Now, since  $J_{n+k} \subseteq J_n$  holds for all  $n \in \mathbb{N}$  and each  $k \geq 0$ , we have  $g_{n+k} \cdot \gamma|_{J_n} \sim_{\circ} \gamma|_{J_n}$  for each such  $n$  and  $k$ . Thus,

- b) If  $\phi: \mathbb{N} \rightarrow \mathbb{N}$  is injective and increasing, the collection  $\{(g_{\phi(n)}, K_n, J_n)\}_{n \in \mathbb{N}}$  still fulfils (15).
- c) If  $\{(g_n, K_n, J_n)\}_{n \in \mathbb{N}}$  fulfils (15), as well as

$$g_n \cdot \gamma(\tau) \in \gamma(J_n) \quad \forall n \in \mathbb{N}, \quad (16)$$

then obviously the same is true for any of its subsequences.<sup>20</sup>

<sup>20</sup>In the following, “passing to a subsequence”, will always mean to replace  $\{(g_n, K_n, J_n)\}_{n \in \mathbb{N}}$  by  $\{(g_{\phi(n)}, K_{\phi(n)}, J_{\phi(n)})\}_{n \in \mathbb{N}}$ , and to redefine  $K := K_{\phi(0)}$ , for some  $\phi: \mathbb{N} \rightarrow \mathbb{N}$  injective and increasing.

Thus,

### Step I

According to Corollary 3.17.1, we can assume that additionally (16), as well as

$$g_{n+1} \cdot \gamma(K_{n+1}) \subseteq \gamma(J_n) \subseteq \gamma(K_n) \subseteq \gamma(K) \quad \forall n \in \mathbb{N} \quad (17)$$

holds. In fact,

- Applying Corollary 3.17.1 to  $L = K_{n+1}$  for  $n \in \mathbb{N}$ , we find  $p(n) \in \mathbb{N}$  with

$$g_p \cdot \gamma(K_p) \subseteq K_{n+1} \quad \forall p \geq p(n).$$

Thus, inductively, we find  $\phi: \mathbb{N} \rightarrow \mathbb{N}$  injective and increasing with

$$g_{\phi(n)} \cdot \gamma(K_{\phi(n)}) \subseteq K_{n+1} \subseteq J_n \quad \text{hence} \quad g_{\phi(n)} \cdot \gamma(\tau) \in J_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Then, by **b)**, replacing  $\{(g_n, K_n, J_n)\}_{n \in \mathbb{N}}$  by  $\{(g_{\phi(n)}, K_n, J_n)\}_{n \in \mathbb{N}}$ , we can assume that (15) and (16) hold right from the beginning.

- Then, applying Corollary 3.17.1 to  $L = K_{n+1}$  for  $n \in \mathbb{N}$ , we find and fix  $p(n) \geq n + 1$  with

$$g_{p(n)} \cdot \gamma(K_{p(n)}) \subseteq K_{n+1} \subseteq J_n.$$

Thus, if we define  $\phi: \mathbb{N} \rightarrow \mathbb{N}$  inductively by  $\phi(0) := 0$  and  $\phi(n) := p(\phi(n-1))$  for  $n \geq 1$ , the subsequence  $\{(g_{\phi(n)}, K_{\phi(n)}, J_{\phi(n)})\}_{n \in \mathbb{N}}$  fulfils (17). Consequently, by **c)**, we can assume that  $\{(g_n, K_n, J_n)\}_{n \in \mathbb{N}}$  fulfils (15), (16) and (17) right from the beginning.  $\ddagger$

Next, observe that for  $J \subseteq I$  an open interval containing  $\tau$ , we find some  $n_0 \in \mathbb{N}$  with  $K_n \subseteq J$  for each  $n \geq n_0$ . Thus, passing to the subsequence, defined by  $\phi: \mathbb{N} \rightarrow \mathbb{N}$ ,  $n \mapsto n + n_0$ , we can replace  $\gamma$  by  $\gamma|_J$ , just by **a)** and **c)**. In particular,

### Step II

We can assume that  $\text{im}[\gamma] \subseteq G \cdot \gamma(\tau)$  holds, because  $\gamma(J) \subseteq G \cdot \gamma(\tau)$  holds for a suitable choice of  $J$ . In fact, by (16), we have  $\lim_n g_n \cdot \gamma(\tau) = \gamma(\tau)$ , as well as  $g_n \cdot \gamma(\tau) \in \text{im}[\gamma]$  for each  $n \in \mathbb{N}$ . Since  $\gamma$  is an embedding, this implies that  $g_n \cdot \gamma(\tau) \in \gamma(t_n)$  holds for some  $t_n \in I \setminus \{\tau\}$  for each  $n \in \mathbb{N}$ , whereby we have  $\lim_n t_n = \tau$ .

Then, applying stability of  $x = \gamma(\tau)$  to  $\{g_n\}_{n \in \mathbb{N}} \subseteq G \setminus G_x$ , we find some  $\{g'_n\}_{n \in \mathbb{N}} \subseteq G \setminus G_x$  with  $\lim_n g'_n = e$ , such that  $g'_n \cdot \gamma(\tau) = \gamma(t_n)$  holds for all  $n \in \mathbb{N}$ . Thus, the claim is clear from Lemma 2.6.  $\ddagger$

Now,

### Step III

For each  $n \in \mathbb{N}$ , we have  $g_n \cdot \gamma|_{J_n} = \gamma \circ \rho_n$  for the analytic diffeomorphism  $\rho_n: J_n \rightarrow I_n \subseteq I$ , given by  $\rho_n := \gamma^{-1} \circ (g_n \cdot \gamma|_{J_n})$ . For this, observe that  $\gamma$  and  $g_n \cdot \gamma|_{J_n}$  are analytic embeddings, and that  $g_n \cdot \gamma(J_n) \subseteq \gamma(J_{n-1}) \subseteq \gamma(I)$  holds by (17). In particular, we have  $I_n \subseteq J_{n-1}$  for each  $n \geq 1$ .

Now, let us say that  $g_n$  is **positive** or **negative** iff  $\dot{\rho}_n > 0$  or  $\dot{\rho}_n < 0$  holds, respectively. Then,

- ▷ If infinitely many  $g_n$  are positive, passing to a subsequence, we can assume that all of them are positive, and that  $g_n \cdot \gamma(K_n) \subseteq K$  holds for all  $n \in \mathbb{N}$  by (17).
- ▷ In the other case, passing to a subsequence, we can assume that each  $g_n$  is negative. Then, (17) shows

$$g_{n+1} \cdot g_{n+2} \cdot \gamma(K_{n+2}) \subseteq g_{n+1} \cdot \gamma(K_{n+1}) \subseteq \gamma(J_n) \quad \forall n \in \mathbb{N},$$

with what (15) and (16) hold for the collection  $\{(g'_n, K'_n, J'_n)\}_{n \in \mathbb{N}}$ , defined by  $K' := K$  and

$$g'_n := g_{n+1} \cdot g_{n+2} \quad K'_n := K_n \quad J'_n := J_n \quad \forall n \in \mathbb{N}.$$



Now, by Corollary 3.17.1, we find  $n_0 \in \mathbb{N}$ , such that  $g'_n \cdot \gamma(K'_n) \subseteq K'$  holds for all  $n \geq n_0$ . Thus, by the same arguments as above, for each  $n \geq n_0$ , we have  $g'_n \cdot \gamma|_{J'_n} = \gamma \circ \rho'_n$  for the analytic diffeomorphism  $\rho'_n: J'_n \rightarrow I'_n$ , defined by  $\rho'_n := \gamma^{-1} \circ (g'_n \cdot \gamma|_{J'_n})$ .

Then, since  $J_{n+2} \subseteq J_n = J'_n$  and  $I_{n+2} \subseteq J_{n+1}$  holds, for  $n \geq n_0$  we obtain

$$\begin{aligned} \rho'_n|_{J_{n+2}} &= \gamma^{-1} \circ (g_{n+1} \cdot g_{n+2} \cdot \gamma|_{J_{n+2}}) = \gamma^{-1} \circ (g_{n+1} \cdot (\gamma \circ \rho_{n+2})) \\ &= \gamma^{-1} \circ ((g_{n+1} \cdot \gamma|_{I_{n+2}}) \circ \rho_{n+2}) = \gamma^{-1} \circ ((g_{n+1} \cdot \gamma|_{J_{n+1}}) \circ \rho_{n+2}) \\ &= \gamma^{-1} \circ ((\gamma \circ \rho_{n+1}) \circ \rho_{n+2}) = \rho_{n+1} \circ \rho_{n+2}, \end{aligned}$$

from which positivity of  $g'_n$  is clear.

We now finally have to show that  $g'_n \notin G_{\gamma(\tau)}$  holds for infinitely many  $n \geq n_0$ . In fact, by **c)**, then we can just pass to a subsequence of  $\{(g'_n, K'_n, J'_n)\}_{n \in \mathbb{N}}$ , in order to achieve that  $g'_n \notin G_{\gamma(\tau)}$  is positive for each  $n \in \mathbb{N}$ .

Now, the above statement follows if we show that  $g'_n \notin G_\gamma$  holds for infinitely many  $n \geq n_0$ . In fact, then Corollary 3.13.1 shows that  $g'_n \in G_{\gamma(\tau)}$  can only hold for finitely many such  $g'_n$ , because  $g'_n \in G_{\gamma(\tau)}$  together with positivity of  $\rho'_n$  implies  $g'_n \cdot \gamma|_{K'_n \cap [\tau, \infty)} \sim_\circ \gamma|_{K'_n \cap [\tau, \infty)}$ .

Thus, assume that  $g'_n \notin G_\gamma$  only holds for finitely many  $n \geq n_0$ , i.e., that there is  $m > n_0$ , such that  $g'_n \in G_\gamma$  holds for all  $n \geq m-1$ . Then, for each  $n \geq m$ , we have  $g_{n+1} = g_n^{-1} \cdot h_n$  for some  $h_n \in G_\gamma$ , hence

$$\begin{aligned} g_{n+2} = g_{n+1}^{-1} \cdot h_{n+1} = h_n^{-1} \cdot g_n \cdot h_{n+1} &\implies g_{n+2} \cdot \gamma(\tau) = h_n^{-1} \cdot g_n \cdot \gamma(\tau) = g_n \cdot \gamma(\tau) \\ &\implies \gamma(\tau) \stackrel{(16)}{=} \lim_n g_{m+2n} \cdot \gamma(\tau) = g_m \cdot \gamma(\tau), \end{aligned}$$

which contradicts that  $g_m \notin G_{\gamma(\tau)}$  holds. For the second equality in the first implication, we have used  $h_n^{-1} \in G_\gamma$ , as well as  $g_n \cdot \gamma(\tau) \in \text{im}[\gamma]$  holds.  $\ddagger$

Thus, we now can assume that  $\text{im}[\gamma] \subseteq G \cdot \gamma(\tau)$  holds; and that we are give a sequence  $\{(g_n, K_n, J_n)\}_{n \in \mathbb{N}}$  which fulfils (15), (16), as well as  $g_n \cdot \gamma(K_n) \subseteq K$ , with  $g_n \in G \setminus G_{\gamma(\tau)}$  in addition positive for each  $n \in \mathbb{N}$ . Then, each of these properties also holds for each subsequence of  $\{(g_n, K_n, J_n)\}_{n \in \mathbb{N}}$ , and the same is true for the property that we will consider now.

#### Step IV

Since  $g_n \notin G_{\gamma(\tau)}$  and  $g_n \cdot \gamma(\tau) \in J_n$  holds, we have

$$g_n \cdot \gamma(\tau) = \gamma(\tau + \Delta_n) \quad \text{for} \quad \tau + \Delta_n \in J_n \subseteq K \quad \text{with} \quad \Delta_n \neq 0.$$

Now,

- ▷ Let us say that  $g_n$  **shifts  $\tau$  to the left** iff  $\Delta_n < 0$  holds, and that  $g_n$  **shifts  $\tau$  to the right** iff  $\Delta_n > 0$ .
- ▷ If  $\Delta_n > 0$  holds for infinitely many  $n \in \mathbb{N}$ , passing to a subsequence, we can assume that each  $g_n$  shifts  $\tau$  to the right, and have done.
- ▷ In the other case, infinitely many  $\Delta_n$  are negative, and we pass to a subsequence, in order to achieve that each  $g_n$  shifts  $\tau$  to the left. Then, we define

$$\gamma' := \gamma \circ \mathbf{i} \quad \tau' := \mathbf{i}^{-1}(\tau) \quad K'_n := \mathbf{i}^{-1}(K_n) \quad J'_n = \mathbf{i}^{-1}(J_n) \quad \rho'_n := \mathbf{i}^{-1} \circ \rho_n \circ \mathbf{i},$$

as well as  $K' := K'_0$  and  $I' := \mathbf{i}(I) = I$ , for the analytic diffeomorphism  $\mathbf{i}: (i', i) = I \rightarrow I$ ,  $t \mapsto i' + i - t$  with  $\mathbf{i}^{-1} = \mathbf{i}$ . Then,

$$I' \supset K' = K'_0 \supset J'_0 \supset K'_1 \supset J'_1 \supset K'_2 \supset J'_2 \supset \dots,$$

as well as  $g_n \cdot \gamma'(K'_n) \subseteq \gamma'(K') \subseteq \gamma'(I')$  and  $\rho'_n: J'_n \rightarrow I'_n \subseteq I'$  holds for  $I'_n = \mathbf{i}^{-1}(I_n) = \mathbf{i}(I_n)$  for all  $n \in \mathbb{N}$ . Now, (15) holds for  $\gamma'$ , because

$$g_n \cdot \gamma'|_{J'_n} = g_n \cdot \gamma \circ \mathbf{i}|_{J'_n} \sim_\circ \gamma \circ \mathbf{i}|_{J'_n} = \gamma'|_{J'_n} \quad \forall n \in \mathbb{N}.$$

In addition to that, (16) holds for  $\gamma'$ , because

$$g_n \cdot \gamma'(\tau') = \gamma(\tau) \in \gamma(J_n) = \gamma'(J'_n).$$

Now,  $\rho'_n$  is obviously positive, and we have  $\gamma'(I') = \gamma(I) \subseteq G \cdot \gamma(\tau) = \gamma'(\tau')$ . Finally,  $g_n \in G \setminus G_{\gamma(\tau)}$  shifts  $\tau'$  to the right, because

$$g_n \cdot \gamma'(\tau') = g_n \cdot \gamma(\tau) = \gamma(\tau + \Delta_n) = \gamma'(\mathbf{i}^{-1}(\tau + \Delta_n)) = \gamma'(\tau' - \Delta_n).$$

Thus, by **a)**, we can proceed with the first case, where each  $g_n$  shifts  $\tau$  to the right.  $\ddagger$

Next, let us modify  $\{g_n\}_{n \in \mathbb{N}}$  in such a way that  $\lim_n g_n = e$  holds.

### Step V

Obviously, we can replace  $\{g_n\}_{n \in \mathbb{N}}$  by  $\{h_n \cdot g_n \cdot h'_n\}_{n \in \mathbb{N}}$  for sequences  $\{h_n\}_{n \in \mathbb{N}}, \{h'_n\}_{n \in \mathbb{N}} \subseteq G_\gamma$ , without affecting any of the properties, we have established so far. Since,  $\text{im}[\gamma] \subseteq G \cdot x$  implies  $G_{[x]} \subseteq G_\gamma$ , and since  $x$  is stable with  $\lim_n g_n \cdot x = x$  by (16), we can modify  $\{g_n\}_{n \in \mathbb{N}}$  in the mentioned way, and then pass to a subsequence, in order to achieve that  $\lim_n g_n = g \in G_x$  exists. We will now show that then already  $g \in G_\gamma$  holds, with what we can replace each  $g_n$  by  $g_n \cdot g^{-1}$ , in order to achieve that  $\lim_n g_n = e$  holds. Now,

- ▷ According to Corollary 3.17.2, we find a compact neighbourhood  $L \subseteq K$  of  $\tau$  and  $n_0 \in \mathbb{N}$ , such that  $g_n \cdot \gamma(L) \subseteq \gamma(K)$  holds for all  $n \geq n_0$ .
- ▷ Then,  $g_n \cdot \gamma|_L = \gamma \circ \tau_n$  holds for some unique analytic diffeomorphism  $\tau_n : L \rightarrow L' \subseteq K$ , just by the same arguments as in the beginning of **Step III**.
- ▷ Let  $n'_0 \geq n_0$  be such that  $J_n \subseteq L$  holds for all  $n \geq n'_0$ . Then, we have  $\dot{\tau}_n > 0$  for each  $n \geq n'_0$ , because  $g_n$  is positive, and since  $\tau_n|_{J_n} = \rho_n$  holds by uniqueness.
- ▷ Then, since  $g_n$  shifts  $\tau$  to the right, for  $L = [l', l]$ ,  $K = [k', k]$ , and  $n \geq n'_0$ , we have  $g_n \cdot \gamma([\tau, l]) \subseteq \gamma([\tau, k])$ .
- ▷ Thus, for each  $t \in [\tau, l]$  and  $n \geq n'_0$ , we have  $g_n \cdot \gamma(t) \in \gamma([\tau, k])$ , hence

$$g \cdot \gamma(t) = \lim_n g_n \cdot \gamma(t) \in \gamma([\tau, k]).$$

Consequently,  $g \cdot \gamma([\tau, l]) \subseteq \gamma([\tau, k])$  holds for  $g \in G_{\gamma(t)}$ , so that  $g \in G_\gamma$  follows from Lemma 2.20.  $\ddagger$

### Step VI

We now can assume that we are given an analytic embedding  $\gamma : I \rightarrow M$ , together with a collection  $\{(g_n, K_n, J_n)\}_{n \in \mathbb{N}}$  fulfilling (15) and (16), such that each  $g_n \in G \setminus G_{\gamma(\tau)}$  is positive, shifts  $\tau$  to the right, and that  $\lim_n g_n = e$  as well as  $g_n \cdot \gamma(K_n) \subseteq K$  holds for each  $n \in \mathbb{N}$ . Since each subsequence of  $\{(g_n, K_n, J_n)\}_{n \in \mathbb{N}}$  has these properties as well, the next lemma applies to each of them.

#### Lemma 3.18

Let  $I \supset K' \supset I' \supset K$  for  $K = [a, b]$ ,  $K' = [a', b']$  compact, and  $I'$  an open interval. Moreover, for each  $n \in \mathbb{N}$ , let  $p(n) \in \mathbb{N}_{>0} \sqcup \{\infty\}$  be maximal with

$$(g_n)^p \cdot \gamma(J_n) = \gamma(I_{n,p}) \quad \forall 0 \leq p \leq p(n),$$

for necessarily unique open intervals  $I_{n,p} = (i'_{n,p}, i_{n,p}) \subseteq K'$ ; hence

$$(g_n)^p \cdot \gamma(\tau) = \gamma(\tau_{n,p}) \quad \text{for } \tau_{n,p} \in I_{n,p} \quad \text{unique} \quad \forall 0 \leq p \leq p(n).$$

Then,

- 1) We have  $p(n) \geq 1$  for each  $n \in \mathbb{N}$ , as well as

$$\tau_{n,p+1} \in I_{n,p} \quad \forall 0 \leq p \leq p(n) - 1 \quad \text{and} \quad \tau = \tau_{n,0} < \tau_{n,1} < \tau_{n,2} < \dots < \tau_{n,p(n)}. \quad (18)$$

- 2) We have  $p(n) < \infty$  for each  $n \in \mathbb{N}$ .

- 3) We have  $b < i_{n,p(n)}$  for infinitely many  $n \in \mathbb{N}$ .  
 4) For each  $n_0 \in \mathbb{N}$ ,  $t \in (\tau, b]$ , and  $\epsilon > 0$  with  $\tau < t - \epsilon$ , we find some  $n \geq n_0$  and  $0 \leq m \leq p(n)$  with  $\tau_{n,m} \in (t - \epsilon, t]$ ; hence,

$$(g_n \cdot h)^k \cdot x \in \gamma((\tau, t]) \quad \forall k = 1, \dots, m \quad \text{and} \quad (g_n \cdot h)^m \cdot x \in \gamma((t - \epsilon, t])$$

for each  $h \in G_\gamma$ .

- 5) If  $q(n) \leq p(n)$  holds for infinitely many  $n \in \mathbb{N}$ , for a sequence  $\{q(n)\}_{n \in \mathbb{N}} \subseteq \mathbb{N}$ , then

$$\lim_n (g_n \cdot h_n)^{q(n)} = g \in G_{\gamma(\tau)} \quad \text{for} \quad \{h_n\}_{n \in \mathbb{N}} \subseteq G_\gamma \quad \implies \quad g \in G_\gamma. \quad (19)$$

PROOF: 1) Since  $g_n \cdot \gamma(J_n) \subseteq \gamma(K) \subseteq \gamma(K')$  holds, we have  $p(n) \geq 1$ . Moreover, the left hand side of (18) is clear from (16), because

$$\gamma(\tau_{n,p+1}) = (g_n)^p \cdot (g_n \cdot \gamma(\tau)) \in (g_n)^p \cdot \gamma(J_n) = \gamma(I_{n,p}) \quad \forall 0 \leq p \leq p(n) - 1.$$

Now, for each  $0 \leq p \leq p(n)$ , let  $\rho_{n,p}: J_n \rightarrow I_{n,p}$  denote the unique analytic diffeomorphism, for which  $(g_n)^p \cdot \gamma|_{J_n} = \gamma \circ \rho_{n,p}$  holds. Then, for the right hand side of (18), let us first show that these diffeomorphisms are positive:

- ▷ Since  $I_{n,0} = J_n$  and  $\rho_{n,0} = \text{id}_{J_n}$  holds,  $\dot{\rho}_{n,0} > 0$  is clear. Moreover,  $\dot{\rho}_{n,1} > 0$  holds by positivity of  $g_n$ , because we have  $\rho_{n,1} = \rho_n: J_n \rightarrow I_{n,1} = I_n$  by uniqueness.
- ▷ Thus, we only have to show that  $\dot{\rho}_{n,p} > 0$  for some  $1 \leq p \leq p(n) - 1$ , implies  $\dot{\rho}_{n,p+1} > 0$ . Now, since  $g_n \cdot \gamma|_{J_n} \sim_\circ \gamma|_{J_n}$  holds by (15), we find  $J \subseteq J_n$  with  $\rho_n(J) \subseteq J_n$ . Thus,

$$\gamma \circ \rho_{n,p+1}|_J = (g_n)^{p+1} \cdot \gamma|_J = (g_n)^p \cdot (g_n \cdot \gamma|_J) = (g_n)^p \cdot (\gamma \circ \rho_n|_J) = \gamma \circ \rho_{n,p} \circ \rho_n|_J,$$

hence  $\rho_{n,p+1}|_J = \rho_{n,p} \circ \rho_n|_J$  holds, from which  $\dot{\rho}_{n,p+1} > 0$  is clear.

Now, since  $g_n$  shifts  $\tau$  to the right,  $\tau_{n,0} = \tau < \tau_{n,1} = \rho_n(\tau) = \tau + \Delta_n \in J_n$  holds for  $\Delta_n > 0$ . Thus, for  $0 \leq p \leq p(n) - 1$ , we get

$$\begin{aligned} \gamma(\tau_{n,p+1}) &= (g_n)^{p+1} \cdot \gamma(\tau) = (g_n)^p \cdot (g_n \cdot \gamma(\tau)) = (g_n)^p \cdot \gamma(\tau_{n,1}) \\ &= (g_n)^p \cdot \gamma(\tau + \Delta_n) = \gamma(\rho_{n,p}(\tau + \Delta_n)) = \gamma(\rho_{n,p}(\tau) + \Delta'_n) = \gamma(\tau_{n,p} + \Delta'_n) \end{aligned} \quad (20)$$

for some  $\Delta'_n > 0$ , because  $\rho_{n,p}$  is positive, and  $\Delta_n > 0$  holds. Thus, the right hand side of (18) follows inductively.

- 2) If  $p(n) = \infty$  holds, then  $\{\tau_{n,p}\}_{p \in \mathbb{N}} \subseteq K'$  is strongly monotonously increasing by Part 1), with limit  $t \in K'$ . Thus, we have

$$\begin{aligned} \gamma(t) = \lim_p \gamma(\tau_{n,p}) &= \lim_p (g_n)^p \cdot \gamma(\tau) \quad \implies \quad \gamma(t) = \lim_p (g_n)^p \cdot (g_n \cdot \gamma(\tau)) \\ &= \lim_p (g_n)^p \cdot \gamma(\tau_{n,1}), \end{aligned}$$

which contradicts that  $\gamma(t)$  is sated, because  $\tau$ ,  $\tau_{n,1}$  and  $t$  are mutually different.

- 3) First observe that  $h_n := (g_n)^{p(n)+1}$  is well defined by Part 2), and that  $h_n \cdot \gamma(K_n) \not\subseteq \gamma(K')$  holds by the definition of  $p(n)$ .

Now, if the statement is wrong, we can pass to a subsequence, in order to achieve that  $i_{n,p(n)} \leq b$  holds for all  $n \in \mathbb{N}$ . Then, it suffices to show that for each  $n \in \mathbb{N}$ , there exists some  $J \subseteq K'$  open with  $J \cap K \neq \emptyset$ , such that

$$(\gamma|_{K'})|_J = (h_n \cdot \gamma)|_{K_n} \circ \rho$$

holds for some analytic diffeomorphism  $\rho: J \rightarrow J'$ . In fact, then for  $t \in J \cap K$ , by Corollary 2.8, we have

$$\begin{aligned} \gamma([a', t]) &\subseteq h_n \cdot \gamma(K_n) \quad \implies \quad \gamma([a', a]) \subseteq h_n \cdot \gamma(K_n) \quad \text{or} \\ \gamma([t, b']) &\subseteq h_n \cdot \gamma(K_n) \quad \implies \quad \gamma([b, b']) \subseteq h_n \cdot \gamma(K_n), \end{aligned}$$

hence  $h_n^{-1} \cdot \gamma([a', a]) \subseteq \gamma(K_n)$  or  $h_n^{-1} \cdot \gamma([b, b']) \subseteq \gamma(K_n)$  for infinitely many  $n \in \mathbb{N}$ , which contradicts satedness of  $\gamma(\tau)$ .

Now, for existence of  $J$ , observe that for  $n \in \mathbb{N}$  fixed

- ▷ We have  $g_n \cdot \gamma(J') \subseteq J_n$  for some open neighbourhood  $J' \subseteq J_n$  of  $\tau$ , just because  $g_n \cdot \gamma(\tau) \in \gamma(J_n)$  holds. Thus,

$$h_n \cdot \gamma(J') = (g_n)^{p(n)} \cdot (g_n \cdot \gamma(J')) \subseteq (g_n)^{p(n)} \cdot \gamma(J_n) = \gamma(I_{n,p(n)}) \subseteq \gamma(K'), \quad (21)$$

with what  $(\gamma|_{K'})|_J = h_n \cdot \gamma \circ \rho$  holds for some analytic diffeomorphism  $\rho: K' \supseteq J \rightarrow J' \subseteq K_n$ .

- ▷ Then,  $h_n \cdot \gamma(\tau) = \gamma(\tau_{n,p(n)+1})$  holds for  $\tau_{n,p(n)+1} := \rho(\tau) < i_{n,p(n)} \leq b$  by (21), and evaluating the right hand side of (20) for  $p = p(n)$ , we also see that  $a < \tau_{n,p(n)} < \tau_{n,p(n)+1}$  holds, hence  $J \cap K \neq \emptyset$ .

- 4) It suffices to prove the statement for  $h = e$ , because  $g^p \cdot \gamma(J_n) \subseteq \text{im}[\gamma]$  holds for  $0 \leq p \leq p(n)$ .

Now, if the statement is wrong, by (18) and Part 3), for infinitely many  $n \geq n_0$ , we have  $(t - \epsilon, t] \subseteq I_{n,q(n)}$  for some  $0 \leq q(n) \leq p(n)$ , hence

$$\gamma((t - \epsilon, t]) \subseteq \gamma(I_{n,q(n)}) = (g_n)^{q(n)} \cdot \gamma(J_n) \implies (g_n)^{-q(n)} \cdot \gamma((t - \epsilon, t]) \subseteq \gamma(J_n),$$

which contradicts that  $\gamma(\tau)$  is sated.

- 5) Passing to a subsequence, we can assume that  $q(n) \leq p(n)$  holds for all  $n \in \mathbb{N}$ . Then, since  $(g_n)^p \cdot \gamma(J_n) \subseteq \text{im}[\gamma]$  holds for  $0 \leq p \leq q(n)$ , for  $g'_n := (g_n \cdot h_n)^{q(n)}$  we have

$$g'_n \cdot \gamma|_{J_n} = (g_n)^{q(n)} \cdot \gamma|_{J_n} = \gamma \circ \rho_{n,q(n)} \quad \forall n \in \mathbb{N}, \quad (22)$$

for  $\rho_{n,q(n)}: J_n \rightarrow I_{n,q(n)} \subseteq K'$  defined as in Part 1). Then,  $\rho_{n,q(n)}$  is positive, and  $g'_n \cdot \gamma(\tau) = \gamma(\tau_{n,q(n)})$  holds for each  $n \in \mathbb{N}$ .

Then,  $\lim_n \gamma(\tau_{n,q(n)}) = \gamma(\tau)$  holds by the left hand side of (19), hence  $\lim_n \tau_{n,q(n)} = \tau$  as  $\gamma$  is an embedding. Passing to a subsequence, we can assume that  $\{\tau_{n,q(n)}\}_{n \in \mathbb{N}}$  is strictly decreasing; and find compact neighbourhoods  $K'_n \subseteq K'$  of  $\tau$  with  $K'_0 := K'$  and

$$\tau_{n,q(n)} \in J'_n := \text{int}[K'_n] \quad K'_{n+1} \subseteq J'_n \quad \forall n \in \mathbb{N} \quad \bigcap_{n \in \mathbb{N}} K'_n = \{\tau\},$$

whereby then (15) holds for  $g'_n$  and  $J'_n$ , because we have  $\tau, \tau_{n,p(n)} \in J'_n$ .

Moreover, each  $g'_n$  shifts  $\tau$  to the right; and by Corollary 3.17.2 (applied to  $\{g'_n, K'_n, J'_n\}_{n \in \mathbb{N}}$ ), we find a compact neighbourhood  $L' \subseteq K'$  of  $\tau$  and  $n_0 \in \mathbb{N}$ , such that  $g'_n \cdot \gamma(L') \subseteq \gamma(K')$  holds for all  $n \geq n_0$ .

Then,  $g'_n \cdot \gamma|_{L'} = \gamma \circ \tau_n$  holds for some unique analytic diffeomorphism  $\tau_n: L' \rightarrow L'' \subseteq K'$ , for each  $n \geq n_0$ . Now,

- ▷ Let  $n'_0 \geq n_0$  be such that  $J_n \subseteq L'$  holds for all  $n \geq n'_0$ . Then,  $\dot{\tau}_n > 0$  holds for each  $n \geq n'_0$ , because  $\tau_n|_{J_n} = \rho_{n,q(n)}|_{J_n}$  holds by (22) and uniqueness.
- ▷ Since each  $g'_n$  shifts  $\tau$  to the right, the rest of the argumentation is now completely analogous to the last two points in **Step V**. ■

As already mentioned above, Lemma 3.18 also applies to each subsequence of  $\{(g_n, K_n, J_n)\}_{n \in \mathbb{N}}$ , and we will tacitly use this in the

**PROOF (OF PROPOSITION 3.14):** Let  $H$  denote the closure in  $G$ , of the group generated by the set  $O_\gamma = \{g \in G \mid g \cdot \gamma \sim_\circ \gamma\}$ . Then,  $G_\gamma \subseteq H$  is a normal subgroup, because  $g^{-1} \cdot q \cdot g \cdot \gamma = \gamma$  holds for each  $q \in G_\gamma$ , and each  $g \in O_\gamma$  by Corollary 2.19. Thus,  $Q := H/G_\gamma$  is a Lie group with Lie algebra  $\mathfrak{q}$ . By general theory, the canonical projection map  $\pi: H \rightarrow Q$  is a Lie group homomorphism with  $\ker[d_e \pi] = \mathfrak{g}_\gamma$ , and we find some smooth local section  $s: V \rightarrow U \subseteq H$ , defined on an open neighbourhood  $V$  of  $[e]$  in  $Q$ , with  $s([e]) = e$  and  $\pi \circ s = \text{id}_V$ , hence  $d_e \pi \circ d_{[e]} s = \text{id}_{\mathfrak{q}}$ .

Let  $\exp$  and  $\exp_{\mathfrak{q}}$  denote the exponential maps of  $G$  and  $Q$ , respectively,<sup>21</sup> and choose an open neighbourhood  $W \subseteq \mathfrak{q}$  of 0 in such a way that  $\exp_{\mathfrak{q}}|_W$  is a homeomorphism to an open neighbourhood  $V' \subseteq V$  of  $[e]$ . Since

<sup>21</sup>Of course,  $\exp|_{\mathfrak{h}}$  is the exponential map of  $H$ , for  $\mathfrak{h}$  the Lie algebra of  $H$ .

$\pi$  is continuous, we have  $\lim_n [g_n] = [e]$ , so that, passing to a subsequence if necessary, we can assume that  $\{[g_n]\}_{n \in \mathbb{N}} \subseteq V'$  holds. Then, for  $\|\cdot\|$  a fixed norm on  $\mathfrak{q}$ , we have

$$[g_n] = \exp_{\mathfrak{q}}(\lambda_n \cdot \vec{q}_n) \quad \text{for some } \vec{q}_n \in \mathfrak{q} \quad \text{with } \|\vec{q}_n\| = 1 \quad \text{and } \lambda_n > 0 \quad \forall n \in \mathbb{N},$$

hence  $\lim_n \lambda_n = 0$ . Then, passing to a subsequence once more, we can assume that  $\lim_n \vec{q}_n = \vec{q} \in \mathfrak{q} \neq 0$  exist, just by compactness of the unit sphere. We define

$$\vec{g} := d_{[e]}s(\vec{q}) \in \mathfrak{h} \subseteq \mathfrak{g} \quad \text{as well as} \quad \vec{g}_n := d_{[e]}s(\vec{q}_n) \in \mathfrak{h} \subseteq \mathfrak{g} \quad \forall n \in \mathbb{N},$$

and observe that then  $\lim_n \vec{g}_n = \vec{g}$  holds by continuity of  $d_{[e]}s$ . Moreover, for each  $n \in \mathbb{N}$ , we have

$$\pi(\exp(\lambda_n \cdot \vec{g}_n)) = \exp_{\mathfrak{q}}(\lambda_n \cdot d_e\pi(\vec{g}_n)) = [g_n] \quad \implies \quad g_n = \exp(\lambda_n \cdot \vec{g}_n) \cdot h_n^{-1} \quad \text{for some } h_n \in H_\gamma,$$

because  $\pi$  is a Lie group homomorphism, and since  $d_e\pi(\vec{g}_n) = \vec{q}_n$  holds. Then,

$$g'_n = g_n \cdot h_n \quad \text{holds for} \quad g'_n := \exp(\lambda_n \cdot \vec{g}_n) \quad \forall n \in \mathbb{N}, \quad (23)$$

so that  $x \rightarrow \gamma(t)$  (cf. Definition 3.2) holds for all  $t \in (\tau, b] \subseteq K = [a, b]$  w.r.t. to the sequence  $\{g'_n\}_{n \in \mathbb{N}}$  by Lemma 3.18.4.

Thus, it remains to show faithfulness of  $\{g'_n\}_{n \in \mathbb{N}}$ , i.e., that  $\vec{g} \notin \mathfrak{g}_x$  holds. For this, it suffices to show that  $\vec{g} \in \mathfrak{g}_x$  implies  $g_t := \exp(t \cdot \vec{g}) \in G_\gamma$  for each  $t \in [0, l]$ , for some  $l > 0$  suitably small; because then  $\vec{g}$  is contained in  $\mathfrak{g}_\gamma$ , which contradicts that  $d_e\pi(\vec{g}) = \vec{q} \neq 0$  holds.

Thus, assume that  $\vec{g} \in \mathfrak{g}_x$  holds, and let  $O$  be a neighbourhood of  $x$  with  $O \cap \gamma((b - \epsilon, b]) = \emptyset$  for some  $\epsilon > 0$ . Then, since  $\varphi_x \circ \exp$  is continuous, and  $\lim_n \vec{g}_n = \vec{g}$  holds, we find  $n_0 \in \mathbb{N}$  and  $l > 0$ , such that the images of the maps  $\delta_n := \gamma_{\vec{g}_n}^x|_{[0, l]}$  are contained in  $O$  for each  $n \geq n_0$ . We fix  $t \in [0, l]$ , and choose  $q(n) \in \mathbb{N}$  maximal with  $q(n) \cdot \lambda_n \leq t$  for each  $n \in \mathbb{N}$ . Since  $\lim_n \lambda_n = 0$  and  $\lim_n \vec{g}_n = \vec{g}$  holds, we have  $t \cdot \vec{g} = \lim_n q(n) \cdot \lambda_n \cdot \vec{g}_n$ , hence

$$G_x \ni g_t = \exp(t \cdot \vec{g}) = \lim_n \exp(q(n) \cdot \lambda_n \cdot \vec{g}_n) = \lim_n (g'_n)^{q(n)} \stackrel{(23)}{=} \lim_n (g_n \cdot h_n)^{q(n)}.$$

Thus,  $g_t \in G_\gamma$  holds by Lemma 3.18.5, provided that  $q(n) \leq p(n)$  holds for infinitely many  $n \in \mathbb{N}$ .

Now, by Lemma 3.18.4, for infinitely many  $n \geq n_0$ , we find some  $m \leq p(n)$  with  $(g_n)^m \cdot x \in \gamma((b - \epsilon, b])$ . But, then  $q(n) < m \leq p(n)$  must hold, because otherwise  $m \cdot \lambda_n \leq q(n) \cdot \lambda_n \leq t \leq l$  implies that

$$(g_n)^m \cdot x = (g'_n)^m \cdot x = \exp(m \cdot \lambda_n \cdot \vec{g}_n) \cdot x = \delta_n(m \cdot \lambda_n) \in \text{im}[\delta_n] \subseteq O$$

holds, which contradicts that  $O \cap \gamma((b - \epsilon, b]) = \emptyset$ . For the second equality, we have used that  $(g_n)^p \cdot x \in \text{im}[\gamma]$  holds for  $0 \leq p \leq m$ .  $\blacksquare$

## 4 Decompositions

In the previous section, we have shown that, if  $\varphi$  is regular, an analytic curve is Lie iff it is not free. In this section, we will show that each free immersive  $\gamma: D \rightarrow M$  is discretely generated by the symmetry group. Roughly speaking, this means that  $\gamma$  can be naturally decomposed into free segments, mutually (and uniquely) related by the group action. For this, it will be sufficient that  $\varphi$  is sated, which we will assume in the following. In addition to that,  $\gamma$  will always denote an analytic immersive curve.

At this point, the reader might recall the statements and notions provided in Subsection 2.3.

### 4.1 Basic properties

To make the above statement a little bit more clear, let  $\gamma: I \rightarrow M$  be free with  $\gamma|_D$  a free segment for some  $D \subset I$ . Moreover, assume that  $g \cdot \gamma|_J = \gamma \circ \rho$  holds for some analytic diffeomorphism  $\rho: D \supseteq J \rightarrow J' \subseteq I \setminus D$ , and some  $g \in G \setminus G_\gamma$ . Then,

$$g \cdot \gamma|_C = \gamma \circ \bar{\rho}|_C \quad \text{holds for} \quad C := D \cap \bar{\rho}^{-1}(I),$$

and we must have  $\overline{\rho}(C) \subseteq I \setminus D$ , since otherwise  $g \cdot \gamma|_D \sim_\circ \gamma|_D$ , hence  $g \in G_{\gamma|_D} = G_\gamma$  holds. Thus, one might ask the question, whether the intervals  $D$  and  $\overline{\rho}(C)$  can be brought together<sup>22</sup> by a suitable choice of  $g$ . Here, it is already clear from the above discussions that this cannot happen, if  $\gamma|_{I'}$  is a free segment for some open interval  $I' \subseteq I$  which contains the closure of  $D$  in  $I$ . Let us take this as a motivation for

**Definition 4.1 (Maximal interval)**

Let  $\gamma: D \rightarrow M$  be an analytic immersion, and  $A \subseteq D$  an interval. Then,  $A$  is called

- free (w.r.t.  $\gamma$ ) iff  $\gamma|_A$  is a free segment.
- maximal (w.r.t.  $\gamma$ ) iff it is free, and iff there exists no free interval  $A' \subseteq D$  properly containing  $A$ .

Obviously, each subinterval of a free interval is free as well; and each maximal  $A$  is necessarily closed in  $D$ , because

$$g \cdot \gamma|_{\overline{A}} \sim_\circ \gamma|_{\overline{A}} \implies g \cdot \gamma|_A \sim_\circ \gamma|_A \implies g \in G_\gamma$$

for  $\overline{A}$  the closure of  $A$  in  $D$ . Moreover,  $A \subseteq D' \subseteq D$  is free w.r.t.  $\gamma|_{D'}$  iff it is free w.r.t. to  $\gamma$ , just because  $G_{\gamma|_{D'}} = G_\gamma$  holds by Lemma 2.18; and it is a straightforward consequence of Zorn's lemma that

**Lemma 4.2**

*If  $\gamma: D \rightarrow M$  is an analytic immersion with  $D' \subseteq D$  free, then we find  $A \subseteq D$  maximal with  $D' \subseteq A$ .*

PROOF: Let  $\mathfrak{D}$  denote the set of all free  $C \subseteq D$  containing  $D'$ . We order  $\mathfrak{D}$  by inclusion, and observe that each chain  $\mathfrak{C}$  in  $\mathfrak{D}$  has the upper bound  $B := \bigcup_{C \in \mathfrak{C}} C$ . In fact,  $B$  is free, because  $g \cdot \gamma|_B \sim_\circ \gamma|_B$  implies  $g \cdot \gamma|_C \sim_\circ \gamma|_C$  for some  $C \in \mathfrak{C}$ , hence  $g \in G_\gamma$  by Lemma 2.18. Thus, by Zorn's lemma, the set of maximal elements in  $\mathfrak{D}$  is non-empty. ■

Now, for a free curve  $\gamma: I \rightarrow M$ , each maximal interval is necessarily closed in  $I = (i', i)$ , hence of the form  $(i', i)$ ,  $(i', \tau]$ ,  $[\tau, i)$  or compact. Of course, in the first case, we have nothing to show because  $I$  is the only maximal interval. Moreover, we will see in Proposition 4.14 that, if  $\gamma$  admits no compact maximal interval, there is  $\tau \in I$  uniquely determined, such that  $(i', \tau]$  and  $[\tau, i)$  are the only maximal intervals, and  $g \cdot \gamma|_{(i', \tau]} \sim_\circ \gamma|_{[\tau, i)}$  holds for some  $g \in G \setminus G_\gamma$ . Finally, we will see that if  $\gamma$  admits some compact maximal interval  $A = [a_-, a_+]$ , there are  $g_-, g_+ \in G \setminus G_\gamma$  and intervals  $A_-, A_+$  closed in  $I$  (and maximal if compact), such that  $g_{\pm 1} \cdot \gamma|_A \sim_\circ \gamma|_{A_{\pm 1}}$  and  $A \cap A_{\pm 1} = \{a_{\pm}\}$  holds. Inductively, then we will construct a decomposition of  $I$  into intervals closed in  $I$ , such that the respective subcurves are related to  $\gamma|_A$  in the same way. To make this precise,

**Definition 4.3**

For  $\gamma: D \rightarrow M$  some fixed analytic curve, define

$$g \sim g' \quad \text{for} \quad g, g' \in G \quad \iff \quad g^{-1} \cdot g' \in G_\gamma.$$

We denote the respective classes by  $[g] = g \cdot G_\gamma$ , and define  $\overline{G}(\gamma)$  to consist of all such classes that are different from  $[e]$ . Observe that then  $\overline{G}(\gamma) = \overline{G}(\gamma|_{D'})$  holds for each  $D' \subseteq D$ , by Lemma 2.18.

Now, let  $\mathfrak{N}$  denote the set of all subsets of  $\mathbb{Z}$ , which are of the form<sup>23</sup>  $\mathfrak{n} = \{n \in \mathbb{Z}_{\neq 0} \mid \mathfrak{n}_- \leq n \leq \mathfrak{n}_+\}$  for  $\mathfrak{n}_-, \mathfrak{n}_+ \in \mathbb{Z}_{\neq 0} \sqcup \{-\infty, \infty\}$  with  $\mathfrak{n}_- < 0 < \mathfrak{n}_+$ . Then, by a decomposition of an interval  $D$ , we will understand a family  $\{a_n\}_{n \in \mathfrak{n}} \subseteq \text{int}[D]$  with  $\mathfrak{n} \in \mathfrak{N}$  and  $a_m < a_n$  if  $m < n$  for  $m, n \in \mathfrak{n}$ . If  $\{a_n\}_{n \in \mathfrak{n}}$  is fixed, we define

$$A_n := [a_{n-1}, a_n] \quad \text{for} \quad \mathfrak{n}_- < n \leq -1 \quad \quad A_0 := [a_{-1}, a_1] \quad \quad A_n := [a_n, a_{n+1}] \quad \text{for} \quad 1 \leq n < \mathfrak{n}_+,$$

as well as  $A_{\mathfrak{n}_-} := D \cap (-\infty, a_{\mathfrak{n}_-}]$  if  $\mathfrak{n}_- \neq -\infty$ , and  $A_{\mathfrak{n}_+} := D \cap [a_{\mathfrak{n}_+}, \infty)$  if  $\mathfrak{n}_+ \neq \infty$  holds. Now,

**Definition 4.4 (Decomposition)**

Let  $\gamma: I \rightarrow M$  be free. Then,

<sup>22</sup>More precisely, this means that the closures of  $D$  and  $\overline{\rho}(C)$  in  $I$ , share exactly one boundary point.

<sup>23</sup>Of course, if  $\mathfrak{n}_- = -\infty$  holds, then  $\mathfrak{n}_- \leq n$  means  $n \in \mathbb{Z}$ , and analogously for  $\mathfrak{n}_+$ .

- 1) For  $\tau \in I = (i', i)$  with  $(i', \tau]$  and  $[\tau, i)$  free, by a  $\tau$ -decomposition of  $\gamma$ , we understand a class  $[g] \neq [e]$ , such that  $g \cdot \gamma|_{(i', \tau]} \rightsquigarrow \gamma|_{[\tau, i)}$  holds w.r.t. the unique analytic diffeomorphism denoted by  $\mu$ .<sup>24</sup>

Then,  $[g]$  is said to be **faithful** iff  $g' \cdot \gamma|_{(i', \tau]} \sim_\circ \gamma$  w.r.t.  $\rho$  implies that either

$$[g'] = [e] \quad \text{and} \quad \bar{\rho}|_{(i', \tau]} = \text{id}_{(i', \tau]} \quad \text{or} \quad [g'] = [g] \quad \text{and} \quad \bar{\rho}|_{\text{dom}[\mu]} = \mu \quad \text{holds.}$$

- 2) For  $A \subseteq I$  compact and free, by an  $A$ -decomposition of  $\gamma$ , we will understand a pair  $(\{a_n\}_{n \in \mathbf{n}}, \{[g_n]\}_{n \in \mathbf{n}})$  with  $\{g_n\}_{n \in \mathbf{n}} \subseteq G$ , and  $\{a_n\}_{n \in \mathbf{n}}$  a decomposition of  $I$ , such that  $A = A_0$ ,  $[g_{\pm 1}] \neq [e]$ , as well as

$$g_n \cdot \gamma|_A \rightsquigarrow \gamma|_{A_n} \quad \forall n \in \mathbf{n} \quad (24)$$

holds. The respective unique analytic diffeomorphisms will be denoted by  $\mu_n$  in the following, and we define  $\mu_0 := \text{id}_A$ , as well as  $g_0 := e$ .

Then,  $(\{a_n\}_{n \in \mathbf{n}}, \{g_n\}_{n \in \mathbf{n}})$  is said to be **faithful** iff

$$g \cdot \gamma|_A \sim_\circ \gamma \quad \text{w.r.t.} \quad \rho \quad \implies \quad [g] = [g_n] \quad \text{and} \quad \bar{\rho}|_{\text{dom}[\mu_n]} = \mu_n \quad \text{for } n \in \mathbf{n} \sqcup \{0\} \text{ unique.}$$

If the  $A$ -decomposition  $(\{a_n\}_{n \in \mathbf{n}}, \{[g_n]\}_{n \in \mathbf{n}})$  is fixed, we define the group elements

$$h_{\pm 1} := g_{\pm 1}, \quad h_n := g_n \cdot g_{n+1}^{-1} \quad \text{for } \mathbf{n}_- \leq n \leq -2, \quad h_n := g_n \cdot g_{n-1}^{-1} \quad \text{for } 2 \leq n \leq \mathbf{n}_+, \quad (25)$$

for which we have

$$h_n \cdot \gamma|_{A_{n+1}} \rightsquigarrow \gamma|_{A_n} \quad \text{for } \mathbf{n}_- \leq n \leq -1 \quad \text{and} \quad h_n \cdot \gamma|_{A_{n-1}} \rightsquigarrow \gamma|_{A_n} \quad \text{for } 1 \leq n \leq \mathbf{n}_+.$$

#### Remark 4.5

Assume that we are in the situation of Definition 4.4.2. Then, as we will see in Lemma 4.6.2,  $[g_{\pm 1}] \neq [e]$  and  $g_{\pm 1} \cdot \gamma|_A \rightsquigarrow \gamma|_{A_{\pm 1}}$  together imply that  $A$  is maximal. Conversely, if we would require  $A$  to be maximal right from the beginning, then  $[g_{\pm 1}] \neq [e]$  would follow from  $g_{\pm 1} \cdot \gamma|_A \rightsquigarrow \gamma|_{A_{\pm 1}}$ .

In fact, let  $A$  be maximal, and  $[g_1] = [e]$ , hence  $\gamma|_A \rightsquigarrow \gamma|_{A_1}$ . Then,  $g \cdot \gamma|_{A \cup A_1} \sim_\circ \gamma|_{A \cup A_1}$  implies that

$$g \cdot \gamma|_A \sim_\circ \gamma|_A \quad \text{or} \quad g \cdot \gamma|_{A_1} \sim_\circ \gamma|_{A_1} \quad \text{or} \quad g \cdot \gamma|_A \sim_\circ \gamma|_{A_1} \quad \text{or} \quad g \cdot \gamma|_{A_1} \sim_\circ \gamma|_A$$

holds; and in each of these cases, we obtain  $g \cdot \gamma|_A \sim_\circ \gamma|_A$  from  $\gamma|_A \rightsquigarrow \gamma|_{A_1}$ . Thus,  $g \in G_\gamma$  holds, which contradicts maximality of  $A$ . The same arguments then also show that  $[g_{-1}] \neq [e]$  must hold.  $\ddagger$

Now, before we are going to construct decompositions explicitly, let us first clarify the following three important facts. First,

#### Lemma 4.6

Let  $\gamma: I \rightarrow M$  be free.

- 1) If  $[g]$  is a  $\tau$ -decomposition of  $\gamma$ , then  $(i', \tau]$  and  $[\tau, i)$  are the only maximal intervals.
- 2) If  $(\{a_n\}_{n \in \mathbf{n}}, \{[g_n]\}_{n \in \mathbf{n}})$  is an  $A$ -decomposition of  $\gamma$ , then  $A$  is maximal, and
  - a) If  $\mathbf{n}_- = -\infty$  holds, then for each  $t \in I$ , we have  $a_n < t$  for some  $n \in \mathbf{n}$ .
  - b) If  $\mathbf{n}_+ = \infty$  holds, then for each  $t \in I$ , we have  $t < a_n$  for some  $n \in \mathbf{n}$ .
- 3) If  $[g]$  is a  $\tau$ -decomposition of  $\gamma$ , there cannot exist any other decomposition of  $\gamma$ .

PROOF: 1) If  $A \subseteq I$  is free, it must be contained in  $(i', \tau]$  or  $[\tau, i)$ , since otherwise  $\tau$  is contained in the interior of  $A$ , so that  $g \cdot \gamma|_{(i', \tau]} \rightsquigarrow \gamma|_{[\tau, i)}$  implies  $g \cdot \gamma|_A \sim_\circ \gamma|_A$ , hence  $[g] = [e]$ .

<sup>24</sup>Observe that the condition  $g \cdot \gamma|_{(i', \tau]} \rightsquigarrow \gamma|_{[\tau, i)}$  already implies  $[g] \neq [e]$ , since  $\gamma$  is injective on a neighbourhood of  $\tau$ . In addition to that, we must have  $g \in G_{\gamma(\tau)}$ , so that  $G_{\gamma(\tau)}$  cannot be trivial, as  $g \neq e$  holds.

- 2) First, it is clear from  $g_{\pm 1} \cdot \gamma|_A \rightsquigarrow \gamma|_{A_{\pm 1}}$  and  $[g_{\pm 1}] \neq [e]$  that  $B$  cannot be free if it properly contains  $A$ , so that  $A$  is maximal.

Second, if a) is wrong, we have  $\lim_{n \rightarrow -\infty} a_n = t$  for some  $i' < t$ , so that for each  $\epsilon > 0$ , we find  $n_\epsilon \in \mathbb{N}$  with  $A_n \subseteq [t, t + \epsilon)$ , hence  $g_n \cdot \gamma(A) \subseteq \gamma([t, t + \epsilon))$  for all  $n \geq n_\epsilon$ . Thus, we find  $\phi: \mathbb{N} \rightarrow \mathbb{N}$  injective and increasing with  $g_{\phi(n)} \cdot \gamma(A) \subseteq \gamma([t, t + 1/n))$ , which contradicts that  $\gamma(t)$  is sated. In the same way, b) follows.

- 3) By the maximality statements in the first two parts, any other decomposition of  $I$  must be a  $\tau$ -decomposition  $[g']$ . But, then  $g \cdot \gamma|_{(i', \tau]} \rightsquigarrow \gamma|_{[\tau, i]}$  and  $g' \cdot \gamma|_{(i', \tau]} \rightsquigarrow \gamma|_{[\tau, i]}$  imply  $g \cdot \gamma|_{(i', \tau]} \sim_\circ g' \cdot \gamma|_{(i', \tau]}$ , hence  $[g'] = [g]$ .  $\blacksquare$

Second,

**Lemma 4.7**

*Each  $\tau$ -decomposition is faithful.*

PROOF: Let  $\delta: I \rightarrow M$  be free with  $\tau$ -decomposition  $[g]$ , and assume that  $g' \cdot \delta|_{(i', \tau]} \sim_\circ \delta$  holds w.r.t.  $\rho: (i', \tau] \supseteq J \rightarrow J'$  for some  $g' \in G$ . Then, we can have

$$g' \cdot \delta|_{(i', \tau]} \sim_\circ \delta|_{(i', \tau]} \implies [g'] = [e] \quad \text{or} \quad g' \cdot \delta|_{(i', \tau]} \sim_\circ \delta|_{[\tau, i]} \implies [g'] = [g].$$

Here, the first implication holds since  $(i', \tau]$  is free, and the second one, because

- ▷ If  $g \cdot \delta|_{(j', \tau]} \rightsquigarrow_{\tau, \tau} \delta|_{[\tau, i]}$  holds for  $i' \leq j' < \tau$ , we have  $g' \cdot \delta|_{(i', \tau]} \sim_\circ g \cdot \delta|_{(i', \tau]}$ , hence  $[g'] = [g]$ .
- ▷ If  $g \cdot \delta|_{(i', \tau]} \rightsquigarrow_{\tau, \tau} \delta|_{[\tau, j]}$  holds for  $\tau < j < i$ , we have

$$g'^{-1} \cdot \delta|_{[\tau, i]} \sim_\circ g^{-1} \cdot \delta|_{[\tau, i]} \implies g' \cdot g^{-1} =: h \in G_\delta \implies g' = g \cdot (g^{-1} \cdot h \cdot g),$$

hence  $[g'] = [g]$  by (7).

In particular, only one of the above cases can occur, since  $[g] \neq [e]$  holds by definition.

- In the first case,  $\delta|_{(i', \tau]} \sim_\circ \delta|_{(i', \tau]}$  holds w.r.t.  $\rho$ , and we have to show that  $\rho$  is the identity on  $J$ . In fact, then  $\bar{\rho}|_{(i', \tau]} = \text{id}_{(i', \tau]}$  is clear from maximality of  $\bar{\rho}$ .

Thus, let us assume that  $\rho \neq \text{id}_J$  holds. Then, applying Lemma 2.12 to  $\gamma = \gamma' := \delta|_{(i', \tau]}$ ,  $B := (i', \tau]$ ,  $\phi := \text{id}|_{(i', \tau]}$ , and  $\psi := \rho$ , we see that  $\delta|_{(i', \tau]}$  is self-related, just because  $J \subseteq B$  and  $\phi|_J \neq \psi$  holds. Thus, by Lemma 2.11, for each  $\epsilon > 0$  suitably small, we have  $\delta|_{(i', \tau]} \sim_\circ \delta|_{(\tau, \tau + \epsilon)}$ , so that

$$g \cdot \delta|_{(i', \tau]} \rightsquigarrow \delta|_{[\tau, i]} \implies g \cdot \delta|_{(i', \tau]} \sim_\circ \delta|_{(i', \tau]} \implies [g] = [e],$$

which contradicts the definitions.

- In the second case,  $g \cdot \delta|_{(i', \tau]} \sim_\circ \delta|_{[\tau, i]}$  holds w.r.t.  $\rho$ , and we have to show that  $\rho$  equals  $\mu$  on  $J$ , as then  $\bar{\rho}|_{\text{dom}[\mu]} = \mu$  is clear from maximality of  $\bar{\rho}$ . Now,

If  $\text{dom}[\mu] = (j', \tau]$  holds for  $i' < j' < \tau$ , we have  $\text{im}[\mu] = [\tau, i]$ , so that  $J \not\subseteq \text{dom}[\mu]$  or  $J \subseteq \text{dom}[\mu]$  and  $\rho \neq \mu|_J$  implies self-relatedness of  $\delta|_{(i', \tau]}$  by Lemma 2.12.<sup>25</sup> Then, the same arguments as in the previous point provide us with a contradiction, so that  $J \subseteq \text{dom}[\mu]$  and  $\rho = \mu|_J$  must hold.

If  $\text{dom}[\mu] = (i', \tau]$  holds, we have  $\text{dom}[\mu^{-1}] = \text{im}[\mu] = [\tau, j]$  for  $\tau < j \leq i$ . Thus,  $J' \not\subseteq \text{dom}[\mu^{-1}]$  or  $J' \subseteq \text{dom}[\mu^{-1}]$  and  $\rho^{-1} \neq \mu^{-1}|_{J'}$  both imply self-relatedness of  $\delta|_{(i', \tau]}$  by Lemma 2.12;<sup>26</sup> with what  $\delta|_{[\tau, i]} \sim_\circ \delta|_{(\tau - \epsilon, \tau]}$  holds for each  $\epsilon > 0$  suitably small, by Lemma 2.11. Then,

$$g \cdot \delta|_{(i', \tau]} \rightsquigarrow \delta|_{[\tau, i]} \implies g \cdot \delta|_{(i', \tau]} \sim_\circ \delta|_{(i', \tau]} \implies [g] = [e],$$

which contradicts the definitions.  $\blacksquare$

<sup>25</sup> Applied to  $\gamma := \delta|_{(i', \tau]}$ ,  $\gamma' := \delta|_{[\tau, i]}$ ,  $B := \text{dom}[\mu]$ ,  $\phi := \mu$ , and  $\psi := \rho$ .

<sup>26</sup> Applied to  $\gamma := \delta|_{[\tau, i]}$ ,  $\gamma' := \delta|_{(i', \tau]}$ ,  $B := \text{im}[\mu] = [\tau, j]$ ,  $\phi := \mu^{-1}$  and  $\psi := \rho^{-1}$ .



Third,

**Lemma 4.8**

If  $\gamma: I \rightarrow M$  is free, each  $A$ -decomposition  $(\{a_n\}_{n \in \mathbb{N}}, \{[g_n]\}_{n \in \mathbb{N}})$  is faithful, so that the reals  $a_n$ , the classes  $[g_n]$ , and the analytic diffeomorphisms  $\mu_n$  are uniquely determined.

PROOF: Let  $g \cdot \gamma|_A \sim_\circ \gamma$  hold w.r.t. the analytic diffeomorphism  $\rho: A \supseteq J \rightarrow J' \subseteq I$ . Then,  $J'$  overlaps some  $A_n$  for  $\mathbf{n}_- \leq n \leq \mathbf{n}_+$  by a) and b) in Lemma 4.6.2, so that we can assume that  $J' \subseteq A_n$  holds. Then,

- ▷  $\text{im}[\mu_n] = A_n$  implies  $g_n \cdot \gamma|_A \sim_\circ g \cdot \gamma|_A$ , hence  $[g] = [g_n]$ .
- ▷ Moreover, if  $J \not\subseteq \text{dom}[\mu_n]$  or  $J \subseteq \text{dom}[\mu_n]$  and  $\rho \neq \mu_n|_J$  holds, then  $\gamma|_A$  is self-related by Lemma 2.12, so that  $\gamma|_A \sim_\circ \gamma|_{A_1}$  holds by Lemma 2.11. Then,  $\text{im}[\mu_1] = A_1$  implies  $g_1 \cdot \gamma|_A \sim_\circ \gamma|_A$ , hence  $[g_1] = [e]$ , which contradicts the definitions.

Thus,  $(\{a_n\}_{n \in \mathbb{N}}, \{[g_n]\}_{n \in \mathbb{N}})$  is faithful, and for the uniqueness statement, we let  $(\{a'_n\}_{n \in \mathbb{N}'}, \{[g'_n]\}_{n \in \mathbb{N}'})$  be another  $A$ -decomposition of  $\gamma$ . Then, we have  $g'_{\pm 1} \cdot \gamma|_A \sim_\circ \gamma|_{A_{\pm 1}}$ , hence  $[g'_{\pm 1}] = [g_{\pm 1}]$  and  $\mu'_{\pm 1} = \mu_{\pm 1}$  by faithfulness. In particular,  $A_{\pm 1} = A'_{\pm 1}$  holds, so that

$$\begin{aligned} \mathbf{n}_- \leq -2 &\implies a_{-2} = a'_{-2} &\implies g'_{-2} \cdot \gamma|_A \sim_\circ \gamma|_{A_{-2}} \\ \mathbf{n}_+ \geq 2 &\implies a_2 = a'_2 &\implies g'_2 \cdot \gamma|_A \sim_\circ \gamma|_{A_2}. \end{aligned}$$

Thus, we can apply the same arguments inductively, in order to conclude that both decompositions, and the corresponding diffeomorphisms coincide. ■

## 4.2 Existence

In the previous subsection, we have investigated the most important properties of decompositions. In this subsection, we will show their existence for compact maximal intervals. At the same time, we will provide the tools to be used in the next subsection, in order to treat the non-compact case. Let us start with the straightforward observation that

**Lemma 4.9**

Let  $\gamma: I \rightarrow M$  and  $\gamma': I' \rightarrow M$  be analytic immersions with  $g \cdot \gamma|_D = \gamma' \circ \rho$  for some  $g \in G$ , and some analytic diffeomorphism  $\rho: I \supseteq D \rightarrow D' \subseteq I'$ .

- 1) If  $D$  is free w.r.t.  $\gamma$ , then  $D'$  is free w.r.t.  $\gamma'$ .
- 2) If  $D$  is compact and maximal w.r.t.  $\gamma$ , then  $D'$  is compact and maximal w.r.t.  $\gamma'$ .

PROOF: 1) For each  $g' \in G$ , we have

$$\begin{aligned} g' \cdot \gamma'|_{D'} \sim_\circ \gamma'|_{D'} &\implies g' \cdot \gamma' \circ \rho \sim_\circ \gamma' \circ \rho &\implies g' \cdot (g \cdot \gamma|_D) \sim_\circ g \cdot \gamma|_D \\ &\implies (g^{-1} \cdot g' \cdot g) \in G_\gamma &\implies g' \cdot g \cdot \gamma \circ \rho^{-1} = g \cdot \gamma \circ \rho^{-1} \\ &&\implies g' \cdot \gamma'|_{D'} = \gamma'|_{D'}. \end{aligned}$$

- 2) If  $D$  is compact,  $\bar{\rho}$  is defined on some open interval  $J \subseteq I$  containing  $D$ . Since  $D' = \bar{\rho}(D)$  is compact and contained in  $I'$ , shrinking  $J$  if necessary, we can assume that  $J' := \bar{\rho}(J) \subseteq I'$  holds. Then,  $\gamma|_J = \gamma' \circ \bar{\rho}|_J$  holds by Lemma 2.1, so that  $\gamma'|_{J'} = \gamma \circ \rho'$  holds for  $\rho' := \bar{\rho}^{-1}|_{J'}$ . Thus, if  $D'$  is not maximal, we find some free interval  $D''$  with  $D' \subset D'' \subseteq J'$ , and then  $\rho'(D'')$  is free (w.r.t.  $\gamma$ ) by the first point, which contradicts maximality of  $D$ . ■

Now,

**Lemma 4.10**

Let  $\gamma: [t', t] \rightarrow M$  be free, and  $[a', a]$  some free interval. Then,

$$a < t \implies [a, k] \text{ is free for some } a < k \leq t, \tag{26}$$

$$t' < a' \implies [k', a'] \text{ is free for some } t' \leq k' < a'. \tag{27}$$

PROOF: First observe that (27) follows from (26) and Lemma 4.9.1, just by replacing  $\gamma$  by  $\gamma \circ i$  for  $i: [t', t] \rightarrow [t', t]$ ,  $s \mapsto t + t' - s$ . Thus, assume that (26) is wrong, and let  $I \subseteq (a', t)$  be an open interval with  $\tau := a \in I$ , on which  $\gamma$  is an embedding. Then,

- ▷ Since  $[a, k]$  is not free for each  $a < k \leq t$ ,  $\gamma|_I$  is continuously generated at  $\tau := a$ , by Corollary 3.13.2.
  - ▷ We choose a collection  $\{(g_n, K_n, J_n)\}_{n \in \mathbb{N}}$  as in the beginning of Subsection 3.3, additionally modified by **Step I** and **Step III**;<sup>27</sup> so that each  $g_n$  is positive.
  - ▷ If  $g_n$  shifts  $\tau$  to the left for some  $n \in \mathbb{N}$ , we have  $g_n \cdot \gamma(J) \subseteq (a', a)$  for some  $J \subseteq J_n \cap (-\infty, a]$ , hence  $g_n \cdot \gamma|_{[a', a]} \sim_{\circ} \gamma|_{[a', a]}$ , which contradicts that  $g_n \notin G_{\gamma(\tau)}$ , hence  $g_n \notin G_{\gamma}$  holds.
  - ▷ If each  $g_n$  shifts  $\tau$  to the right, we apply Corollary 3.17.2, in order to fix some compact neighbourhood  $L \subseteq I$  of  $\tau$ , and some  $n_0 \in \mathbb{N}$ , such that  $g_n \cdot \gamma(L) \subseteq \gamma(I)$  holds for all  $n \geq n_0$ .
  - ▷ We write  $L = [l', l]$ , and conclude from positivity of  $g_n$ , that  $g_n \cdot \gamma([l', \tau]) = \gamma([l'_n, \tau_n])$  holds for  $[l'_n, \tau_n] \subseteq I$  with  $g_n \cdot \gamma(\tau) = \gamma(\tau_n)$  and  $g_n \cdot \gamma(l') = \gamma(l'_n)$  for each  $n \geq n_0$ .
  - ▷ If  $l'_n < \tau$  holds for some  $n \in \mathbb{N}$ , we have  $g_n \cdot \gamma|_{[a', a]} \sim_{\circ} \gamma|_{[a', a]}$ , and obtain a contradiction as above.
- In the other case,  $g_n \cdot \gamma([l', \tau]) \subseteq \gamma([\tau, \tau_n])$  holds for all  $n \in \mathbb{N}$ , which contradicts that  $\varphi$  is sated, as  $\lim_n \gamma(\tau_n) = \lim_n g_n \cdot \gamma(\tau) = \gamma(\tau)$  holds by (16). ■

Next, let us consider the situation where, in the above lemma, the interval  $[a', a]$  is in addition maximal.

**Proposition 4.11**

Let  $\gamma: [t', t] \rightarrow M$  be free, and  $[a', a]$  maximal. Then,

- 1) If  $a < t$  holds, there exists  $[g] \in \overline{G}(\gamma)$  uniquely determined by  $g \cdot \gamma|_{[a', a]} \sim_{\circ} \gamma|_{[a, k]}$  for all  $a < k \leq t$ , and we either have

$$g \cdot \gamma|_{[a', s]} \rightsquigarrow_+ \gamma|_{[a, s']} \quad \text{for some} \quad s \leq a < s' \quad \text{or} \quad (28)$$

$$g \cdot \gamma|_{[s, a]} \rightsquigarrow_- \gamma|_{[a, s']} \quad \text{for some} \quad s < a < s'. \quad (29)$$

- 2) If  $t' < a'$  holds, there exists  $[g'] \in \overline{G}(\gamma)$  uniquely determined by  $g' \cdot \gamma|_{[a', a]} \sim_{\circ} \gamma|_{[k', a']}$  for all  $t' \leq k' < a'$ , and we either have

$$g' \cdot \gamma|_{[s, a]} \rightsquigarrow_+ \gamma|_{[s', a']} \quad \text{for some} \quad s' < a' \leq s \quad \text{or} \quad (30)$$

$$g' \cdot \gamma|_{[a', s]} \rightsquigarrow_- \gamma|_{[s', a']} \quad \text{for some} \quad s' < a' < s. \quad (31)$$

*Comment:*

- ▷ Obviously, (28) just means that  $g$  “right shifts” the initial segment  $\gamma|_{[a', s]}$  to  $\gamma|_{[a, s']}$ , and (29) that  $g$  “flips” the final segment  $\gamma|_{[s, a]}$  at  $\gamma(a)$  to  $\gamma|_{[a, s']}$ .
- ▷ Analogously, (30) means that  $g'$  “left shifts” the final segment  $\gamma|_{[s, a]}$  to  $\gamma|_{[s', a']}$ , and (31) that  $g'$  “flips” the initial segment  $\gamma|_{[a', s]}$  at  $\gamma(a')$  to  $\gamma|_{[s', a']}$ .

PROOF: It suffices to show the first part, as the second one then follows from Lemma 4.9, just by replacing  $\gamma$  by  $\gamma \circ i$  for  $i: [t', t] \rightarrow [t', t]$ ,  $s \mapsto t + t' - s$ . Now,

- If (28) or (29) holds, then  $q \cdot \gamma|_{[a', a]} \sim_{\circ} \gamma|_{[a, k]}$  for each  $a < k \leq t$ , implies  $q \cdot \gamma|_{[a', a]} \sim_{\circ} g \cdot \gamma|_{[a', a]}$ , hence  $[q] = [g]$ , which shows the uniqueness statement.
- For the “either or statement”, observe that if (28) and (29) hold at the same time, we have  $\gamma|_{[a', r]} \rightsquigarrow_- \gamma|_{[s, a]}$  for some reals  $a' < r, s < a$ , which contradicts Corollary 2.9.

For existence of  $[g]$ , let us shrink  $t$  in such a way that  $\gamma|_{[a, t]}$  is a free segment (cf. Lemma 4.10), and an embedding; and let  $\{t_n\}_{n \in \mathbb{N}} \subseteq (a, t]$  be strongly decreasing with  $\lim_n t_n = a$ . Since, by maximality of  $[a', a]$ , the intervals  $[a', t_n]$  are not free, we find  $\{g_n\}_{n \in \mathbb{N}} \subseteq G \setminus G_{\gamma}$  with  $g_n \cdot \gamma|_{[a', t_n]} \sim_{\circ} \gamma|_{[a', t_n]}$  for each  $n \in \mathbb{N}$ . Then,

<sup>27</sup>Observe that, for the respective modifications, only satedness of  $\varphi$  has been used, and that the modifications in **Step II** are irrelevant for the modifications in **Step III**.

- ▷ We have  $g_n \cdot \gamma|_{[a',a]} \sim \gamma|_{[a,t_n]}$  or  $g_n \cdot \gamma|_{[a,t_n]} \sim \gamma|_{[a',a]}$  for each  $n \in \mathbb{N}$ , because  $[a',a]$  and  $[a,t_n]$  are free.
- ▷ Thus, replacing  $g_n$  by  $g_n^{-1}$  if necessary, we can assume that for each  $n \in \mathbb{N}$ , we have

$$\gamma|_{[a,t_n]} \sim \gamma_n \cdot \gamma|_{[a',a]} \implies (\gamma|_{[a',t]})|_{J_n} = g_n \cdot \gamma|_{[a',a]} \circ \rho_n$$

for some analytic diffeomorphism  $\rho_n: [a, t_n] \supseteq J_n \rightarrow J'_n \subseteq [a', a]$ .

- ▷ Now, for each  $n \in \mathbb{N}$ , let us define

$$[c'_n, c_n] = C_n := [a', t] \cap \bar{\rho}_n^{-1}([a', a]) \quad \text{as well as} \quad L_n := \bar{\rho}_n(C_n) \subseteq [a', a].$$

- ▷ Then, for each  $n \in \mathbb{N}$ , we have

$$(\gamma|_{[a',t]})|_{C_n} = g_n \cdot \gamma|_{[a',a]} \circ \bar{\rho}_n|_{C_n} \implies c'_n \in [a, t_n]. \quad (32)$$

In fact, by the left hand side,  $c'_n < a$  would imply  $\gamma|_{[a',a]} \sim g_n \cdot \gamma|_{[a',a]}$ , hence  $g_n \in G_\gamma$ ; and  $c'_n \leq t_n$  must hold, because we necessarily have  $c'_n \leq \inf[J_n]$ .

Now,

### Case I

If  $\bigcup_{n \in \mathbb{N}} \{[g_n]\}$  is finite, passing to a subsequence, we can assume that  $[g_n] = [g]$  holds all  $n \in \mathbb{N}$ , for some  $[g] \in \bar{G}(\gamma)$ . Then, (32) gives (recall that we have  $a \leq c'_n$ )

$$(\gamma|_{[a,t]})|_{C_n} = g \cdot \gamma|_{[a',a]} \circ \bar{\rho}_n|_{C_n} \quad \forall n \in \mathbb{N}. \quad (33)$$

In particular, we have  $a' < c'_n$ , hence  $\bar{\rho}_n(c'_n) \in \{a', a\}$  by (5). Thus, passing to a subsequence, we can assume that  $\bar{\rho}_n(c'_n) =: b \in \{a', a\}$  holds for all  $n \in \mathbb{N}$ . Since  $\gamma|_{[a,t]}$  is injective with  $c'_n \in [a, t_n] \subseteq [a, t]$ , we have

$$\gamma|_{[a,t]}(c'_n) \stackrel{(33)}{=} g \cdot \gamma(b) \quad \forall n \in \mathbb{N} \implies c'_0 = c'_n \quad \forall n \in \mathbb{N} \xrightarrow{\lim_n t_n = a} c'_0 = a.$$

Thus,  $a = c'_0$  and  $\bar{\rho}_0(a) = \bar{\rho}_0(c'_0) = b$  holds, so that

- ▷ If  $b = a'$  holds, we have  $\bar{\rho}_0(a) = a'$ , hence  $\gamma|_{[a,c_0]} = g \cdot \gamma|_{[a',\bar{\rho}_0(c_0)]} \circ \bar{\rho}_0|_{[a,c_0]}$  by (33), showing (28).
- ▷ If  $b = a$  holds, we have  $\bar{\rho}_0(a) = a$ , hence  $\gamma|_{[a,c_0]} = g \cdot \gamma|_{[\bar{\rho}_0(c_0),a]} \circ \bar{\rho}_0|_{[a,c_0]}$  by (33), showing (29).

### Case II

If  $\bigcup_{n \in \mathbb{N}} \{[g_n]\}$  is infinite, passing to a subsequence, we can assume that  $[g_n] \neq [g_m]$  holds for all  $\mathbb{N} \ni n \neq m \in \mathbb{N}$ . Then, neither  $C_n \cap [a', a]$  nor  $C_n \cap C_m$  for  $n \neq m$  can contain some open interval, because

- ▷ If  $J \subseteq C_n \cap C_m$  is non-empty and open such that  $\gamma|_J$  is an embedding, we have

$$\begin{aligned} (g_n \cdot \gamma)(\bar{\rho}_n(J)) &= \gamma(J) = (g_m \cdot \gamma)(\bar{\rho}_m(J)) &\implies g_m^{-1} \cdot g_n \cdot \gamma|_{[a',a]} &\sim \gamma|_{[a',a]} \\ & &\implies g_m^{-1} \cdot g_n &\in G_\gamma, \end{aligned}$$

hence  $m = n$ , by the choice of the sequence  $\{g_n\}_{n \in \mathbb{N}}$ .

- ▷ Similarly,  $J \subseteq C_n \cap [a', a]$  implies  $g_n \cdot \gamma|_{[a',a]} \sim \gamma|_{[a',a]}$ , hence  $[g_n] = [e]$ , which contradicts the choices.

Thus, we have  $a \leq c'_n < c_n$  for all  $n \in \mathbb{N}$ , as well as either

$$c'_n < c_n \leq c'_m < c_m \quad \text{or} \quad c'_m < c_m \leq c'_n < c_n \quad \text{for some} \quad \mathbb{N} \ni n \neq m \in \mathbb{N}. \quad (34)$$

Then, even  $a < c'_n$  must hold, because for each  $n \in \mathbb{N}$ , we have  $a \leq c'_m < t_m < c_n$  for some  $m \in \mathbb{N}$ , hence  $C_m \subseteq [a, c'_n]$  by the right hand side of (34).

▷ Thus, for each  $n \in \mathbb{N}$ , we find  $p(n) \in \mathbb{N}$  with  $t_{p(n)} < c'_n$ , hence

$$c'_{p(n)} < t_{p(n)} < c'_n \quad \xrightarrow{(34)} \quad C_{p(n)} \subseteq [a, c'_n].$$

▷ Then, for  $\phi: \mathbb{N} \rightarrow \mathbb{N}$ , inductively defined by  $\phi(0) := 0$  and  $\phi(n) := p(\phi(n-1))$  for  $n \geq 1$ , we have  $C_{\phi(n+1)} \subseteq [a, c'_{\phi(n)}]$ , hence

$$g_{\phi(n+1)} \cdot \gamma(L_{\phi(n+1)}) = \gamma(C_{\phi(n+1)}) \subseteq \gamma([a, c'_{\phi(n)}]) \quad \text{and} \quad a < c'_{\phi(n)} < t_{\phi(n)} \quad \forall n \in \mathbb{N}, \quad (35)$$

i.e.,  $\lim_n c'_{\phi(n)} = a$  by the right hand side.

▷ Now,  $L_{\phi(n)} = [a', a]$  must hold for  $n \geq 1$ , because otherwise  $c'_{\phi(n)} = a'$  or  $c_{\phi(n)} = t$  must hold by (5), which contradicts that both  $[a', a] \cap C_{\phi(n)}$  and  $C_{\phi(n)} \cap C_{\phi(0)}$  have non-empty interior.

▷ Thus, the left hand side of (35) reads  $g_{\phi(n+1)} \cdot \gamma([a', a]) \subseteq \gamma([a, c'_{\phi(n)}])$ , which contradicts that  $\gamma(a)$  is sated, as  $\lim_n c'_{\phi(n)} = a$  holds. ■

Now, let  $\gamma: I \rightarrow M$  be free, and  $[a', a] \subseteq (i', i) = I$  be maximal. Then, applying the above proposition to  $\gamma|_{[t', t]}$  for some  $i' < t' < a' < a < t < i$ , we conclude that either (28) or (29), and either (30) or (31) holds. Thus, it is clear from the discussion in Subsection 2.3 that

$$\begin{aligned} (28) & \implies \text{we either have} & g \cdot \gamma|_{[a', j']} \rightsquigarrow_+ \gamma|_{[a, i]} & \text{or} & g \cdot \gamma|_{[a', a]} \rightsquigarrow_+ \gamma|_{[a, b]}, \\ (29) & \implies \text{we either have} & g \cdot \gamma|_{[j, a]} \rightsquigarrow_- \gamma|_{[a, i]} & \text{or} & g \cdot \gamma|_{[a', a]} \rightsquigarrow_- \gamma|_{[a, b]}, \\ (30) & \implies \text{we either have} & g' \cdot \gamma|_{[j, a]} \rightsquigarrow_+ \gamma|_{[i', a']} & \text{or} & g' \cdot \gamma|_{[a', a]} \rightsquigarrow_+ \gamma|_{[b', a']}, \\ (31) & \implies \text{we either have} & g' \cdot \gamma|_{[a', j']} \rightsquigarrow_- \gamma|_{[i', a']} & \text{or} & g' \cdot \gamma|_{[a', a]} \rightsquigarrow_- \gamma|_{[b', a']}, \end{aligned}$$

for some  $a' < j', j < a$  and  $i' < b' < a' < a < b < i$ . Now, in the cases on the right hand side, the intervals  $[a, b]$  and  $[b', a']$  are maximal by Lemma 4.9.2, so that we can apply the same arguments inductively, in order to conclude that

#### Corollary 4.12

If  $\gamma: I \rightarrow M$  is free, each compact maximal interval  $A \subseteq I$  admits an  $A$ -decomposition of  $\gamma$ .

PROOF: Applying the above arguments inductively, we obtain a decomposition  $\{a_n\}_{n \in \mathbb{N}}$  of  $I$  with  $A_0 = A$ , as well as elements  $h_n \in G \setminus G_\gamma$  with

$$h_n \cdot \gamma|_{A_{n+1}} \rightsquigarrow \gamma|_{A_n} \quad \text{for all } \mathbf{n}_- \leq n \leq -1 \quad \text{and} \quad h_n \cdot \gamma|_{A_{n-1}} \rightsquigarrow \gamma|_{A_n} \quad \text{for all } 1 \leq n \leq \mathbf{n}_+.$$

Then,  $(\{a_n\}_{n \in \mathbb{N}}, \{[g_n]\}_{n \in \mathbb{N}})$  is the desired  $A$ -decomposition of  $\gamma$ , provided that we define  $g_n := h_n \cdot \dots \cdot h_{-1}$  for  $\mathbf{n}_- \leq n \leq -1$ , as well as  $g_n := h_n \cdot \dots \cdot h_1$  for  $1 \leq n \leq \mathbf{n}_+$ . ■

### 4.3 Non-compact decompositions

In the previous subsection, we have shown existence of decompositions for compact maximal intervals. In this brief subsection, we will use Lemma 4.10 and Proposition 4.11, in order to show that a free curve either admits a compact maximal interval or a (necessarily unique)  $\tau$ -decomposition. For this, let us first show that

#### Lemma 4.13

Let  $\delta: I \rightarrow M$  be free, and  $[a', a]$  maximal w.r.t.  $\gamma := \delta|_{[a', t]}$  for some  $t > a$  or maximal w.r.t.  $\gamma := \delta|_{[t', a]}$  for some  $t' < a'$ . Then,  $[a', a]$  is maximal w.r.t.  $\delta$  if (28) or (30) holds w.r.t.  $\gamma$ , respectively.

PROOF: We only show the first case, as the second one follows analogously. Thus, let  $[a', a]$  be maximal w.r.t.  $\gamma := \delta|_{[a', t]}$ , and assume that (28) holds. Then,  $[a', a]$  is maximal w.r.t.  $\delta$ , because

▷ By (28), for  $0 < \epsilon$  suitably small, we have  $g \cdot \delta|_{[a', a]} \rightsquigarrow \delta|_{[a', a+\epsilon]}$ , so that  $[a', a+\epsilon]$  cannot be free.

▷ Combining (28) with Lemma 2.3, we find  $r, r' > 0$  with  $g \cdot \delta|_{[a'-r', a']} \rightsquigarrow_+ \delta|_{[a-r, a]}$ , hence  $g \cdot \delta|_{[a'-\epsilon, a]} \rightsquigarrow \delta|_{[a', a]}$  for each  $0 < \epsilon \leq r'$ , so that  $[a'-\epsilon, a]$  cannot be free as well. ■

Then, we easily conclude that

**Proposition 4.14**

*If  $\delta: I \rightarrow M$  is free but not a free segment, it either admits a  $\tau$ -decomposition or some compact maximal interval.*

PROOF: First, it is clear from Lemma 4.6.3 that only one of the mentioned situations can hold. Thus, we have to show that non-existence of a compact maximal interval implies the existence of a  $\tau$ -decomposition of  $\delta$ . Now, if  $\delta$  admits no compact maximal interval, there must exist some  $\tau \in I$ , such that  $(i', \tau]$  or  $[\tau, i)$  is maximal, just because  $I$  is not free by assumption. Then,

- ▷ If  $[\tau, i)$  is maximal, we fix some  $i' < t' < \tau < t < i$ , and observe that  $[\tau, t]$  is free. Then, applying Lemma 4.10 to  $\gamma := \delta|_{[t', t]}$ , we see that  $[s, \tau]$  is free for some  $t' < s < \tau$ , and choose  $A$  maximal w.r.t.  $\delta$  with  $[s, \tau] \subseteq A$ .
- ▷ Then,  $A$  must be of the form  $(i', \tau']$  for some  $\tau \leq \tau'$ , since  $I$  is not free, since  $\delta$  admits no compact maximal interval, and because  $A$  cannot be of the form  $[\tau'', i)$  for  $\tau'' \leq s$ , by maximality of  $[\tau, i)$ .
- ▷ Thus, we can assume that  $(i', \tau]$  is maximal right from the beginning, and then the above arguments (now applied to  $(i', \tau]$  instead of  $[\tau, i)$ ) show that  $[\tau', i)$  is maximal for some  $\tau' \leq \tau$ , i.e., that  $[\tau, i)$  is free.

Thus,  $[a', \tau]$  and  $[\tau, t]$  are free for each  $i' < a' < \tau < t < i$ , and we now have to show that then already  $g \cdot \gamma|_{(i', \tau]} \rightsquigarrow \gamma|_{[\tau, i)}$  must hold for some  $g \in G \setminus G_\delta$ . For this, let  $\tau < t < i$  be fixed. Then,

- ▷ If  $[a', \tau]$  is maximal w.r.t.  $\gamma := \delta|_{[a', t]}$  for some  $i' < a' < \tau$ , then (29) must hold for  $a = \tau$  by Lemma 4.13, because  $[a', \tau]$  is not maximal w.r.t.  $\delta$  as it is compact. Thus, we find  $g \in G \setminus G_\delta$ , and  $s < \tau < s'$  with

$$g \cdot \delta|_{[s, \tau]} \rightsquigarrow_{\tau, \tau} \delta|_{[\tau, s']} \quad \text{hence} \quad g \cdot \delta|_{(i', \tau]} \rightsquigarrow \delta|_{[\tau, i)}.$$

- ▷ If  $[a', \tau]$  is not maximal w.r.t.  $\delta|_{[a', t]}$ , for each  $i' < a' < \tau$ , we choose  $\{a'_n\}_{n \in \mathbb{N}} \subseteq (i', \tau)$  decreasing with  $\lim_n a'_n = i'$ .

Then, for each  $n \in \mathbb{N}$ , we choose  $A_n \subseteq [a'_n, t]$  maximal w.r.t.  $\delta|_{[a'_n, t]}$  with  $[a'_n, \tau] \subseteq A_n$ ; with what  $A_n = [a'_n, t_n]$  holds for some  $\tau < t_n \leq t$ . If  $t_n = t$  holds for all  $n \in \mathbb{N}$ , then  $(i', t] = \bigcup_{n \in \mathbb{N}} A_n$  is free (cf. proof of Lemma 4.2), which contradicts maximality of  $(i', \tau]$ . Thus,  $[a', a]$  is maximal w.r.t.  $\gamma := \delta|_{[a', t]}$  for some  $a' < \tau < a < t$ , so that (29) holds by Lemma 4.13. Then,  $\tau < a < t$  implies  $g \cdot \gamma|_{[\tau, t]} \sim_\circ \gamma|_{[\tau, t]}$  for  $[g] \neq [e]$ , which contradicts that  $[\tau, t]$  is free. ■

**Example 4.15**

Let  $G = \text{SO}(2)$  act via rotations<sup>28</sup> on  $M = \mathbb{R}^2$ , and consider the analytic immersion

$$\delta: (-\infty, \infty) \rightarrow \mathbb{R}^2, \quad t \mapsto (t, t^3).$$

Then, for  $g_\pi \in \text{SO}(2)$  the rotation by  $\pi$ , as well as  $(i', \tau] = (-\infty, 0]$ , we have

$$g_\pi \cdot \delta|_{(-\infty, 0]} \rightsquigarrow \delta|_{[0, \infty)} \quad \text{w.r.t.} \quad \rho: (-\infty, 0] \rightarrow [0, \infty), \quad t \mapsto -t.$$

Here,  $(-\infty, 0]$  and  $[0, \infty)$  are free, because  $g \cdot \delta(t) = \delta(t')$  holds for  $t, t' \neq 0$  and  $g \in \text{SO}(2)$  iff we have  $g = g_\pi$  and  $t' = -t$  or  $g = \text{id}_{\mathbb{R}^2}$  and  $t = t'$ . Then, maximality of these intervals is clear, because

$$g_\pi \cdot \delta|_{(-\infty, \epsilon]} \sim_\circ \delta|_{(-\infty, \epsilon]} \quad \text{and} \quad g_\pi \cdot \delta|_{[-\epsilon, \infty)} \sim_\circ \delta|_{[-\epsilon, \infty)}$$

holds for each  $\epsilon > 0$ . ‡

Moreover, combining Corollary 4.12 with Proposition 4.14, we easily obtain

**Corollary 4.16**

*If  $\gamma: D \rightarrow M$  is a self-related free segment, so is  $\bar{\gamma}: I \rightarrow M$ .*

<sup>28</sup>The corresponding action is sated by a) in Remark 2.15.2, and even regular, because it is proper by compactness of  $\text{SO}(2)$ .

PROOF: If the statement is wrong,  $\bar{\gamma}$  either admits a  $\tau$ -decomposition or an  $A$ -decomposition with  $D \subseteq A$ . In the first case,  $D$  is contained in  $(i', \tau]$  or  $[\tau, i)$  by Lemma 4.6.1, so that  $e \cdot \gamma|_{(i', \tau]} \sim_{\circ} \gamma|_{[\tau, i)}$  holds by Lemma 2.11, which contradicts faithfulness. In the second case, since  $D \subseteq A$  holds, Lemma 2.11 shows  $e \cdot \gamma|_A \sim_{\circ} \gamma|_{A_1}$ , contradicting faithfulness as well.  $\blacksquare$

For instance, if  $M = \mathbb{C} \cong \mathbb{R}^2$  holds, then  $\gamma: t \mapsto e^{it}$  is a free segment iff  $\gamma|_{(-\epsilon, 2\pi)}$  is a free segment for each  $\epsilon > 0$ .

**Corollary 4.17**

*If  $\gamma: D \rightarrow M$  is free and  $D' \subset D$  maximal, then  $\gamma|_{D'}$  is not self-related.*

PROOF: If  $\gamma|_{D'}$  is self-related, then  $D$  is free by Corollary 4.16, which contradicts maximality of  $D'$ .  $\blacksquare$

#### 4.4 The compact case

In this subsection, we will investigate the case where the free curve  $\gamma: I \rightarrow M$  admits some compact maximal interval  $B = [b', b]$  in more detail.

For this, let  $B_-, B_+ \subseteq I$  be intervals that are closed in  $I$ , with  $g_- \cdot \gamma|_B \rightsquigarrow \gamma|_{B_-}$  and  $g_+ \cdot \gamma|_B \rightsquigarrow \gamma|_{B_+}$  for  $\sup(B_-) = b'$  and  $\inf(B_+) = b$ , as well as  $[g_{\pm}] \neq [e]$ .<sup>29</sup> Then, we have

$$\begin{aligned} g_+ \cdot \gamma|_B \rightsquigarrow_+ \gamma|_{B_+} &\iff g_- \cdot \gamma|_B \rightsquigarrow_+ \gamma|_{B_-}, \\ g_+ \cdot \gamma|_B \rightsquigarrow_- \gamma|_{B_+} &\iff g_- \cdot \gamma|_B \rightsquigarrow_- \gamma|_{B_-} \end{aligned} \quad (36)$$

holds. In fact, the second line is clear from the first one, just by faithfulness of the  $B$ -decomposition of  $\gamma$ ; and the first line follows from the more general statement that

$$g_+ \cdot \gamma|_B \rightsquigarrow_+ \gamma|_{B_+} \implies g_- \cdot \gamma|_B \rightsquigarrow_+ \gamma|_{B_-} \quad \text{for } [g_-] = [g_+^{-1}], \quad (37)$$

$$g_- \cdot \gamma|_B \rightsquigarrow_+ \gamma|_{B_-} \implies g_+ \cdot \gamma|_B \rightsquigarrow_+ \gamma|_{B_+} \quad \text{for } [g_+] = [g_-^{-1}]. \quad (38)$$

For this observe that  $g_+ \cdot \gamma|_B \rightsquigarrow_+ \gamma|_{B_+}$  implies ((38) follows analogously)

$$g_+^{-1} \cdot \gamma|_{[b, b+\delta]} \rightsquigarrow_{b, b'} \gamma|_{[b', b'+\delta']} \xrightarrow{\text{Lemma 2.3}} g_+^{-1} \cdot \gamma|_{[b-\epsilon, b]} \rightsquigarrow_{b, b'} \gamma|_{[b'-\epsilon', b']}$$

for some  $\delta, \delta', \epsilon, \epsilon' > 0$  suitably small, hence (37) by faithfulness.  $\dagger$

Thus, if  $\gamma$  has the  $A$ -decomposition  $(\{a_n\}_{n \in \mathbb{N}}, \{[g_n]\}_{n \in \mathbb{N}})$ , then we either have

$$g_{\pm 1} \cdot \gamma|_A \rightsquigarrow_+ \gamma|_{A_{\pm 1}} \quad \text{or} \quad g_{\pm 1} \cdot \gamma|_A \rightsquigarrow_- \gamma|_{A_{\pm 1}},$$

and we will say that  $A$  is positive/negative iff the **left/right** hand side of the above equation holds. Then, if we define  $\rightsquigarrow_{-n} := \rightsquigarrow_-$  iff  $|n|$  is odd, and  $\rightsquigarrow_{-n} := \rightsquigarrow_+$  if  $|n|$  is even, we get

**Lemma 4.18**

*Let  $\gamma: I \rightarrow M$  be free with  $A$ -decomposition  $(\{a_n\}_{n \in \mathbb{N}}, \{[g_n]\}_{n \in \mathbb{N}})$ . Then,<sup>30</sup>*

*1) If  $A$  is positive/negative, each compact  $A_n$  is positive/negative, so that we have*

$$h_n \cdot \gamma|_{A_{n+1}} \rightsquigarrow_{+/-} \gamma|_{A_n} \quad \text{for } \mathbf{n}_- \leq n \leq -1 \quad \text{and} \quad h_n \cdot \gamma|_{A_{n-1}} \rightsquigarrow_{+/-} \gamma|_{A_n} \quad \text{for } 1 \leq n \leq \mathbf{n}_+.$$

*Consequently, for each  $n \in \mathbb{N}$ , we have*

$$g_n \cdot \gamma|_A \rightsquigarrow_+ \gamma|_{A_n} \quad \text{if } A \text{ is positive} \quad \text{and} \quad g_n \cdot \gamma|_A \rightsquigarrow_{-n} \gamma|_{A_n} \quad \text{if } A \text{ is negative.}$$

*2) If  $A$  is negative, each free interval is contained in some  $A_n$  for  $\mathbf{n}_- \leq n \leq \mathbf{n}_+$ , so that these intervals are the only maximal ones.*

<sup>29</sup>Thus,  $B_{\pm}$  are the intervals  $B_{\pm 1}$  that correspond to the unique  $B$ -decomposition of  $\gamma$ , whereby  $g_{\pm} \in [g_{\pm 1}]$  holds.

<sup>30</sup>Confer Definition 4.4.2, for the definition of the quantities  $h_n$ .

3) If  $A$  is positive/negative, each compact maximal interval is positive/negative.

PROOF: 1) Each  $A_n$  is free, and each compact one even maximal by Lemma 4.9. Thus, the first statement follows inductively from faithfulness of each  $A_n$ -decomposition for  $\mathbf{n}_- < n < \mathbf{n}_+$ , equation (36), and

$$\begin{aligned} g_{+/-} \cdot \gamma|_B \rightsquigarrow_+ \gamma|_{B_{+/-}} \quad \text{for } B_{+/-} \text{ compact} &\implies g_{+/-}^{-1} \cdot \gamma|_{B_{+/-}} \rightsquigarrow_+ \gamma|_B, \\ g_{+/-} \cdot \gamma|_B \rightsquigarrow_- \gamma|_{B_{+/-}} \quad \text{for } B_{+/-} \text{ compact} &\implies g_{+/-}^{-1} \cdot \gamma|_{B_{+/-}} \rightsquigarrow_- \gamma|_B. \end{aligned}$$

The second statement is then clear from

$$g_n = h_n \cdot \dots \cdot h_{-1} \quad \forall \mathbf{n}_- \leq n \leq -1 \quad \text{as well as} \quad g_n = h_n \cdot \dots \cdot h_1 \quad \forall 1 \leq n \leq \mathbf{n}_+.$$

2) By Part 1), each compact  $A_n$  is negative. Thus,  $a_n$  cannot be contained in the interior of some free interval for each  $n \in \mathbf{n}$ , just because  $g \cdot \gamma|_{[a_n - \epsilon, a_n]} \rightsquigarrow_{a_n, a_n} \gamma|_{[a_n, a_n + \epsilon']}$  holds for some  $g \in G \setminus G_\gamma$  and  $\epsilon, \epsilon' > 0$ . Thus, the claim is clear from Lemma 4.6.2.

3) If  $A$  is negative, each compact maximal interval is negative by Part 1), just because it equals some compact  $A_n$  by Part 2). Thus, there cannot exist any negative interval if  $A$  is positive and vice versa. ■

Now, let us say that  $\gamma$  is **positive/negative** iff it admits some positive/negative interval, with what each other compact maximal interval is positive/negative by Lemma 4.18.3. Thus,

**Proposition 4.19**

If  $\gamma: I \rightarrow M$  is positive, there is  $[h] \in \overline{G}(\gamma)$  unique, such that for each positive  $A \subseteq I$  with corresponding decomposition  $(\{a_n\}_{n \in \mathbf{n}}, \{[g_n]\}_{n \in \mathbf{n}})$ , we have  $[h_n] = [h^{\text{sign}(n)}]$  for each  $n \in \mathbf{n}$ , hence

$$[g_n] = [h^n] \quad \forall n \in \mathbf{n} \quad \implies \quad h^n \cdot \gamma|_A \rightsquigarrow_+ \gamma|_{A_n} \quad \forall n \in \mathbf{n}. \quad (39)$$

Moreover, for each  $t \in I$ , there is some positive interval  $A_t$ , such that  $t$  is contained in the interior of  $A_t$ . More precisely, we have

- For  $\mathbf{n}_- < n \leq 0$  and  $b' \in \text{int}[A_{n-1}]$ , we find  $b \in \text{int}[A_n]$ , such that  $[b', b]$  is positive.
- For  $0 \leq n < \mathbf{n}_+$  and  $b \in \text{int}[A_{n+1}]$ , we find  $b' \in \text{int}[A_n]$ , such that  $[b', b]$  is positive.

PROOF: If such a class  $[h]$  exists, it is necessarily unique by faithfulness. Moreover, if  $A \subseteq I$  is positive with respective  $A$ -decomposition  $(\{a_n\}_{n \in \mathbf{n}}, \{[g_n]\}_{n \in \mathbf{n}})$ , the implication in (39) is clear from the last equation in Lemma 4.18.1. Now, for this decomposition, let us define  $h := g_1$ , and observe that

▷ Since  $h_1 = g_1 = h$  holds, we can assume that  $[h_n] = [h^{\text{sign}(n)}]$  holds for some  $1 \leq n < \mathbf{n}_+$ . Then,  $h_n \cdot \gamma|_{A_{n-1}} \rightsquigarrow_+ \gamma|_{A_n}$  (Lemma 4.18.1) implies

$$h_n^{-1} \cdot \gamma|_{A_n} \rightsquigarrow_+ \gamma|_{A_{n-1}} \xrightarrow{(38)} h_n \cdot \gamma|_{A_n} \rightsquigarrow_+ \gamma|_{A_{n+1}} \implies [h_{n+1}] = [h_n] = [h],$$

so that  $[h_n] = [h^{\text{sign}(n)}]$  follows inductively for all  $1 \leq n \leq \mathbf{n}_+$ .

▷ Since  $[h_{-1}] = [g_{-1}] \stackrel{(37)}{=} [g_1^{-1}] = [h^{-1}]$  holds, we can argue in the same way, in order to conclude that  $[h_n] = [h^{\text{sign}(n)}]$  also holds for  $\mathbf{n}_- \leq n \leq -1$ .

From this, the left hand side of (39) follows easily, because

▷ We have  $[g_1] = [h]$ ; and if  $[g_n] = [h^n]$  holds for some  $1 \leq n < \mathbf{n}_+$ , we get

$$[g_{n+1}] = [h_{n+1} \cdot g_n] = [h \cdot q \cdot h^n] = [h^{n+1} \cdot h^{-n} \cdot q \cdot h^n] \stackrel{(7)}{=} [h^{n+1}]$$

for some  $q \in G_\gamma$  with  $h_{n+1} = h \cdot q$ , so that the claim follows inductively for all  $1 \leq n \leq \mathbf{n}_+$ .

▷ Then, since  $[h_{-1}] = [h^{-1}]$  holds, the statement follows in the same way for all  $\mathbf{n}_- \leq n \leq -1$ .

Now, let  $B = [b', b]$  be positive with  $B$ -decomposition  $(\{b_n\}_{n \in \mathbf{n}'}, \{g'_n\}_{n \in \mathbf{n}'})$ . We have to show that then  $[h'] = [h]$  holds for  $h' := g'_1$ . For this, first observe that

- i)  $B$  can neither be contained in  $A_{\mathbf{n}_-}$  if  $\mathbf{n}_- \neq \infty$  holds nor be contained in  $A_{\mathbf{n}_+}$  if  $\mathbf{n}_+ \neq \infty$  holds, just because these intervals are free and not compact, and because  $B$  is compact maximal.
- ii) If  $B \subseteq A_n$  or  $A_n \subseteq B$  holds for some  $\mathbf{n}_- < n < \mathbf{n}_+$ , we must have  $B = A_n$  by maximality of  $B$  and  $A_n$ .

Now, if  $B = A_n$  holds for some  $\mathbf{n}_- < n < \mathbf{n}_+$ , then  $[h'] = [h]$  is clear from faithfulness, and  $h \cdot \gamma|_B = h \cdot \gamma|_{A_n} \rightsquigarrow_+ \gamma|_{A_{n+1}}$ . In the other case, i) and ii) show that there must exist some compact  $A_n = [a', a]$ , such that either

$$b' < a' < b < a \quad \text{or} \quad a' < b' < a < b \quad \text{holds.}$$

Then, we have

$$\begin{array}{llll} b' < a' < b < a & \xrightarrow{\text{ii)}} & B_1 = [b, i] & \text{or} \quad B_1 = [b, \beta] \quad \text{for } b < a < \beta \\ a' < b' < a < b & \xrightarrow{\text{ii)}} & A_{n+1} = [a, i] & \text{or} \quad A_{n+1} = [a, \alpha] \quad \text{for } a < b < \alpha, \end{array}$$

from which  $[h] = [h']$  is clear, because

- In the first case,  $h \cdot \gamma|_{A_n} \rightsquigarrow \gamma|_{A_{n+1}}$  implies  $h \cdot \gamma|_B \sim_\circ \gamma|_{B_1}$ , hence  $[h] = [g'_1] = [h']$  by faithfulness.
- In the second case,  $h' \cdot \gamma|_B \rightsquigarrow \gamma|_{B_1}$  implies  $h' \cdot \gamma|_A \sim_\circ \gamma|_{A_1}$ , hence  $[h'] = [g_1] = [h]$  by faithfulness.

Finally, it remains to show that

- ▷ For  $\mathbf{n}_- < n \leq 0$  and  $b' \in \text{int}[A_{n-1}]$ , we find  $b \in \text{int}[A_n]$ , such that  $[b', b]$  is maximal.
- ▷ For  $0 \leq n < \mathbf{n}_+$  and  $b \in \text{int}[A_{n+1}]$ , we find  $b' \in \text{int}[A_n]$ , such that  $[b', b]$  is maximal.

Here, we will only show the second statement, as the first one follows analogously. For this, let  $\mu$  denote the unique analytic diffeomorphism that corresponds to the relation  $h \cdot \gamma|_{A_n} \rightsquigarrow_+ \gamma|_{A_{n+1}}$ . Then, for  $b \in \text{int}[A_{n+1}]$ ,

$$h \cdot \gamma|_{A_n} \rightsquigarrow_+ \gamma|_{A_{n+1}} \quad \text{w.r.t. } \mu \quad \implies \quad h \cdot \gamma|_{[a_n, b']} \rightsquigarrow_+ \gamma|_{[a_{n+1}, b]} \quad \text{w.r.t. } \mu|_{[a_n, b']} \quad (40)$$

for  $b' = \mu^{-1}(b) \in \text{int}[A_n]$ . Then, it is immediate from (40) and  $a_n < b' < a_{n+1} < b$  that  $B := [b', b]$  is maximal iff it is free. Now, for  $g \in G \setminus G_\gamma$ , we have

$$\begin{array}{llll} g \cdot \gamma|_B \sim_\circ \gamma|_B & \implies & g^p \cdot \gamma|_{[b', a_{n+1}]} \sim_\circ \gamma|_{[a_{n+1}, b]} & \text{w.r.t. } \rho \\ & \implies & g^p \cdot \gamma|_{A_n} \sim_\circ \gamma|_{A_{n+1}} & \text{w.r.t. } \rho \\ & \implies & h \cdot \gamma|_{A_n} \rightsquigarrow_+ \gamma|_{A_{n+1}} & \text{w.r.t. } \bar{\rho}|_{A_n} = \mu \end{array} \quad (41)$$

for  $p \in \{-1, 1\}$ . Here, the first implication holds, because  $[b', a_{n+1}]$  and  $[a_{n+1}, b]$  are free, and the last one is clear from faithfulness of the decomposition that corresponds to the positive interval  $A_n$ . Now, by the last line in (41) and the definition of  $\rho$ , we have  $\mu(t) \in (a_{n+1}, b)$  for some  $t \in (b', a_{n+1})$ . This, however, contradicts that  $\mu$  is positive as  $\mu(b') = b$  holds. ■

#### Example 4.20

Let  $G = \mathbb{R}$  act via  $\varphi(t, (x, y)) := (t + x, y)$  on  $M = \mathbb{R}^2$ , and define  $\gamma: \mathbb{R} \rightarrow M$ ,  $t \mapsto (t, \sin(t))$ .

- ▷ Then,  $\varphi$  is regular as it is pointwise proper, hence sated. The interval  $[t, t + 2\pi]$  is positive for each  $t \in \mathbb{R}$ , and the class  $[h]$  is given by  $[2\pi]$ .
- ▷ If we replace  $\gamma$  by its restriction to  $(0, \infty)$ , it admits the non-compact maximal interval  $(0, 2\pi]$ . But,  $\gamma$  is not a free segment, and there cannot exist any  $\tau$ -decomposition of  $\gamma$ , just because  $G_x = \{e\}$  holds for each  $x \in M$ . Thus, Proposition 4.14 shows that there must be some compact maximal interval, which is indeed the case for  $[t, t + 2\pi]$  for each  $t > 0$ . ‡

Thus, it remains to discuss the situation where  $\gamma: I \rightarrow M$  is negative. In this case, the classes  $[g_{-1}]$  and  $[g_1]$  need not to be related in any way. Indeed,



**Example 4.21**

Let  $G = \mathbb{R}^2 \rtimes \text{SO}(2)$  be the euclidean group acting on  $\mathbb{R}^2$  in the canonical way, and  $\gamma: \mathbb{R} \ni t \mapsto (t, \sin(t))$ .

- ▷ Then,  $\varphi$  is regular as it is pointwise proper, hence sated. Moreover, the interval  $A = [0, \pi]$  is negative, and  $[g_{-1}]$  and  $[g_1]$  are classes of the rotations by  $\pi$  around  $(0, 0)$  and  $(\pi, 0)$ , respectively.
- ▷ If we restrict  $\gamma$  to  $(-\pi, \pi)$ , it obviously admits the  $\tau$ -decomposition  $[g_{-1}]$  for  $\tau = 0$ . ‡

Anyhow, each  $[g_n]$  can be expressed in terms of the elements  $[g_{-1}]$  and  $[g_1]$ . More precisely, for  $\sigma: \mathbb{Z}_{\neq 0} \rightarrow \{-1, 1\}$ , defined by

$$\sigma(n) := \begin{cases} (-1)^{n-1} & \text{if } n > 0 \\ (-1)^n & \text{if } n < 0, \end{cases} \quad (42)$$

we have

**Proposition 4.22**

If  $\gamma: I \rightarrow M$  is negative with  $A$ -decomposition  $(\{a_n\}_{n \in \mathbb{N}}, \{[g_n]\}_{n \in \mathbb{N}})$ , then the intervals  $\{A_n\}_{\mathbb{N} \leq n \leq \mathbb{N}_+}$  are the only maximal ones, and we have

$$g_n \cdot \gamma|_A \rightsquigarrow_{-n} \gamma|_{A_n} \quad \text{with} \quad [g_n] = [g_{\sigma(\text{sign}(n))}] \cdots [g_{\sigma(n)}] \quad \forall n \in \mathbb{N}. \quad (43)$$

In fact, by the first two parts of Lemma 4.18, it remains to verify (43); for which we let  $\gamma: I \rightarrow M$  be a fixed free curve, and define  $O_\gamma := \{g \in G \mid g \cdot \gamma \sim_\circ \gamma\}$ . Then, let us first observe that

- a) If  $h \cdot \gamma|_{[a-\epsilon, a]} \rightsquigarrow_{a, a} \gamma|_{[a, a+\epsilon']}$  holds for  $h \in G \setminus G_\gamma$  and  $\epsilon, \epsilon' > 0$ , we have  $[h] = [h^{-1}]$  if  $[a, a + \epsilon']$  is free.

In fact, by Lemma 2.3,  $h \cdot \gamma|_{[a, a+\delta]} \rightsquigarrow_{a, a} \gamma|_{[a-\delta', a]}$  holds for some  $0 < \delta \leq \epsilon'$ , and some  $0 < \delta' < \epsilon$ , so that

$$h \cdot \gamma|_{[a, a+\delta]} \rightsquigarrow_{a, a} \gamma|_{[a-\delta', a]} \rightsquigarrow_{a, a} h^{-1} \cdot \gamma|_{[a, a+\delta'']} \quad \text{holds for some} \quad 0 < \delta'' < \epsilon',$$

hence  $[h] = [h^{-1}]$ , because  $[a, a + \epsilon']$  is free.

- b) Let  $A_- := [a_-, a]$  and  $A_+ := [a, a_+]$  be negative with

$$h \cdot \gamma|_{A_-} \rightsquigarrow_{-} \gamma|_{A_+} \quad \stackrel{\text{a)}}{\iff} \quad h \cdot \gamma|_{A_+} \rightsquigarrow_{-} \gamma|_{A_-} \quad \text{for some} \quad h \in G \setminus G_\gamma.$$

Moreover, let  $A_{--}, A_{++} \subseteq I$  be closed in  $I$  with  $A_{--} \cap A_- = \{a_-\}$ ,  $A_{++} \cap A_+ = \{a_+\}$ , as well as

$$h_- \cdot \gamma|_{A_-} \rightsquigarrow_{-} \gamma|_{A_{--}} \quad \text{and} \quad h_+ \cdot \gamma|_{A_+} \rightsquigarrow_{-} \gamma|_{A_{++}} \quad \text{for some} \quad h_\pm \in G \setminus G_\gamma.$$

Then, we have  $[h_-] = [h \cdot h_+ \cdot h]$  and  $[h_+] = [h \cdot h_- \cdot h]$ , because

- ▷ By Lemma 2.3,  $h \cdot \gamma|_{A_-} \rightsquigarrow_{a, a} \gamma|_{A_+}$  implies  $h \cdot \gamma|_{[a_--\epsilon, a_-]} \rightsquigarrow_{a_-, a_+} \gamma|_{[a_+, a_++\epsilon']}$  for some  $\epsilon, \epsilon' > 0$ .

- ▷ Combining this with  $(h_- \cdot h) \cdot \gamma|_{A_+} \rightsquigarrow \gamma|_{A_{--}}$  and  $h_+ \cdot \gamma|_{A_+} \rightsquigarrow \gamma|_{A_{++}}$ , we see that

$$h_+ \cdot \gamma|_{A_+} \sim_\circ (h \cdot h_- \cdot h) \cdot \gamma|_{A_+} \implies [h_+] = [h \cdot h_- \cdot h] \quad \text{as} \quad A_+ \quad \text{is free.}$$

- ▷ Thus, we find  $q \in G_\gamma$  with

$$h_+ \cdot q = h \cdot h_- \cdot h \implies [h_-] = [h \cdot q' \cdot h_+ \cdot q \cdot h] \stackrel{(7)}{=} [h \cdot q' \cdot (h_+ \cdot h)] \stackrel{(7)}{=} [h \cdot h_+ \cdot h]$$

for  $q' \in G_\gamma$  with  $h^{-1} = h \cdot q'$ . Here, for the last equality, we have used that  $(h_+ \cdot h) \cdot \gamma|_{A_-} \rightsquigarrow \gamma|_{A_{++}}$ , hence  $(h_+ \cdot h) \in O_\gamma$  holds.

In particular, if we are in the situation of Proposition 4.22, then a) shows that

$$[h_n] = [h_n^{-1}] \quad \forall n \in \mathbb{N} \quad \text{hence} \quad [g_{\pm 1}] = [h_{\pm 1}] = [h_{\pm 1}^{-1}] = [g_{\pm 1}^{-1}], \quad (44)$$

because each compact  $A_n$  is negative. In particular, since  $g_{\pm 1}^{-1} \in O_\gamma$  holds, for  $q \in G_\gamma$  and  $n \in \mathbb{N}$ , we have

$$[g_{\pm 1} \cdot q \cdot g_{\pm 1}] \stackrel{(44)}{=} [g_{\pm 1} \cdot q \cdot g_{\pm 1}^{-1}] \stackrel{(7)}{=} [e] \implies q_n^\pm := (g_{\mp 1} \cdot g_{\pm 1})^n \cdot (g_{\pm 1} \cdot g_{\mp 1})^n \in G_\gamma, \quad (45)$$

which follows inductively. Moreover, b) shows that

$$\begin{aligned} \mathbf{n}_- \leq -2: \quad [h_{n-1}] &= [h_n \cdot h_{n+1} \cdot h_n] & \forall \mathbf{n}_- < n \leq -2 & \quad \text{as well as} & \quad [h_{-2}] &= [g_{-1} \cdot g_1 \cdot g_{-1}], \\ \mathbf{n}_+ \geq 2: \quad [h_{n+1}] &= [h_n \cdot h_{n-1} \cdot h_n] & \forall 2 \leq n < \mathbf{n}_+ & \quad \text{as well as} & \quad [h_2] &= [g_1 \cdot g_{-1} \cdot g_1] \end{aligned} \quad (46)$$

holds, and we finally observe that

$$\mathbf{n}_- \leq -2 \implies g_1 \cdot g_{-1} \in O_\gamma \quad \text{as well as} \quad \mathbf{n}_+ \geq 2 \implies g_{-1} \cdot g_1 \in O_\gamma. \quad (47)$$

In fact, if  $\mathbf{n}_+ \geq 2$  holds, we have (the case  $\mathbf{n}_- \leq -2$  follows analogously)

$$\begin{aligned} g_1 \cdot \gamma|_{A_0} \rightsquigarrow_- \gamma|_{A_1} & \stackrel{\text{Lemma 2.3}}{\implies} g_1 \cdot \gamma|_{A_{-1}} \sim_\circ \gamma|_{A_2} & \stackrel{(44)}{\implies} \gamma|_{A_{-1}} \sim_\circ g_1 \cdot \gamma|_{A_2} \\ & \implies g_{-1} \cdot \gamma|_{A_{-1}} \sim_\circ (g_{-1} \cdot g_1) \cdot \gamma & \stackrel{(44)}{\implies} \gamma \sim_\circ (g_{-1} \cdot g_1) \cdot \gamma, \end{aligned}$$

whereby in the last step, we have used that  $g_{-1} \cdot \gamma|_A \rightsquigarrow_- \gamma|_{A_{-1}}$  implies  $\gamma|_A \rightsquigarrow_- g_{-1} \cdot \gamma|_{A_{-1}}$  by (44).  $\ddagger$

We now are ready for the

PROOF (OF PROPOSITION 4.22): We have to show the right hand side of (43), for which we first verify that

$$[h_n] = [g_{-1} \cdot (g_1 \cdot g_{-1})^{|\mathbf{n}|-1}] \quad \forall \mathbf{n}_- \leq n \leq -1 \quad \text{and} \quad [h_n] = [g_1 \cdot (g_{-1} \cdot g_1)^{n-1}] \quad \forall 1 \leq n \leq \mathbf{n}_+ \quad (48)$$

holds. This is clear for  $n = \pm 1$ , as well as, by the right hand side of (46), for  $n = -2$  and  $n = 2$  if  $\mathbf{n}_- \leq -2$  and  $\mathbf{n}_+ \geq 2$  holds, respectively.

Thus, if  $\mathbf{n}_+ \geq 3$  (the case  $\mathbf{n}_- \leq -3$  follows analogously), we can assume that (48) holds for all  $1 \leq n \leq m$  for some  $2 \leq m < \mathbf{n}_+$ , and argue by induction:

Let us first observe that  $h_{m-1} \cdot h_m \in O_\gamma$  holds, because

$$[h_{m-1} \cdot h_m] \stackrel{(48)}{=} [g_1 \cdot q_{m-2}^+ \cdot h_2] \stackrel{(7),(45)}{=} [g_1 \cdot h_2] = [g_1^2 \cdot (g_{-1} \cdot g_1)] \stackrel{(7)}{=} [g_{-1} \cdot g_1] \quad (49)$$

as  $g_1^2 \in G_\gamma$  holds by (45), and since  $(g_{-1} \cdot g_1) \in O_\gamma$  holds by (47). Thus, we have

$$[h_{m+1}] \stackrel{(46)}{=} [h_m \cdot (h_{m-1} \cdot h_m)] \stackrel{(49)}{=} [h_m \cdot (g_{-1} \cdot g_1)] \stackrel{(7)}{=} [g_1 \cdot (g_{-1} \cdot g_1)^m],$$

which shows the claim.

Then, the right hand side of (43) follows inductively from (48). In fact, this formula obviously holds for  $n = \pm 1$ , so that we can assume that it holds for all  $1 \leq n \leq m$  for some  $1 \leq m < \mathbf{n}_+$  (the other direction follows in the same way). Then, if  $m = 2 \cdot k$  is even, we have

$$[g_{m+1}] = [h_{m+1} \cdot g_m] \stackrel{(43),(48),(7)}{=} [g_1 \cdot (g_{-1} \cdot g_1)^{2k} \cdot (g_1 \cdot g_{-1})^k] = [g_1 \cdot (g_{-1} \cdot g_1)^k \cdot q_k^+] \stackrel{(45)}{=} [g_{\sigma(1)} \cdot \dots \cdot g_{\sigma(m+1)}].$$

Similarly, if  $m = 2k + 1$  is odd, we have

$$\begin{aligned} [g_{m+1}] &= [h_{m+1} \cdot g_m] \stackrel{(43),(48),(7)}{=} [g_1 \cdot (g_{-1} \cdot g_1)^{2k+1} \cdot (g_1 \cdot g_{-1})^k \cdot g_1] \\ &= [(g_1 \cdot g_{-1})^{k+1} \cdot g_1 \cdot q_k^+ \cdot g_1] \stackrel{(45)}{=} [(g_1 \cdot g_{-1})^{k+1}] = [g_{\sigma(1)} \cdot \dots \cdot g_{\sigma(m+1)}]. \end{aligned} \quad \blacksquare$$

Thus, we finally obtain

#### Theorem 4.23

*If  $\gamma: I \rightarrow M$  is free but not a free segment, it either admits a unique  $\tau$ -decomposition or a compact maximal interval. In the second case,  $\gamma$  is either positive or negative, with what the statements in Proposition 4.19 or Proposition 4.22 hold, respectively.*

Here, “unique” has to be understood in the way that the real  $\tau \in I$ , as well as the corresponding  $\tau$ -decomposition is unique, cf. Lemma 4.6.3.

**Corollary 4.24**

*If  $\varphi$  is sated and free, then a free curve  $\gamma: I \rightarrow M$  is either a free segment or positive. Thus, for each  $t \in I$ , we find  $J \subseteq I$  free with  $t \in J$ , such that  $g \cdot \gamma(J) \cap \gamma(J)$  is finite for all  $g \neq e$ .*

PROOF: Since  $\varphi$  is free, it only admits trivial stabilizers, so that no  $\tau$ -decomposition, and no negative interval can exist. Then, for each  $t \in I$ , we find  $J' \subseteq I$  free with  $t \in J'$ . This is clear if  $\gamma$  is a free segment, and follows from Proposition 4.19 if  $\gamma$  is positive. Now, let us shrink  $J'$  around  $t$  in such a way that  $\gamma|_{J'}$  is an embedding, and choose a compact neighbourhood  $K \subseteq J'$  of  $t$ . Then, if  $g \cdot \gamma(J) \cap \gamma(J)$  is infinite for  $J := \text{int}[K]$ , Lemma 2.3 shows that  $g \cdot \gamma|_K \sim_\circ \gamma|_K$ , hence  $g \in G_\gamma = \{e\}$  holds. ■

By the last statement in Remark 2.15.2, the above corollary in particular applies to the situation where a Lie group acts by left multiplication on itself.

## 4.5 Arbitrary domains

To this point, we only have discussed decompositions of free curves  $\gamma: D \rightarrow M$  with  $D$  an open interval. This was mainly for technical reasons, because if  $D$  is open, there are no ambiguities concerning the conventions we have fixed in the end of Subsection 2.3 when defining decompositions of free curves.<sup>31</sup> However, the general case now follows easily from the statements we have proven so far, just by considering the maximal analytic extension  $\bar{\gamma}: \bar{I} \rightarrow M$  of  $\gamma$ , as well as the restriction  $\underline{\gamma} := \gamma|_I$  of  $\gamma$  to  $(i', i) = I := \text{int}[D]$ . Indeed, the key observation then is that  $\bar{\gamma}$  and  $\underline{\gamma}$  are free as well, and that

**Lemma 4.25**

*$\gamma$  is a free segment iff  $\underline{\gamma}$  is a free segment.*

PROOF: It is clear that  $\underline{\gamma}$  is a free segment if  $\gamma$  is so. Now, if  $\underline{\gamma}$  is a free segment, then  $I$  is free w.r.t.  $\gamma$ , so that  $D$  is free w.r.t.  $\gamma$ , as it is the closure of  $I$  in  $D$ . ■

Thus, Theorem 4.23 immediately provides us with

**Corollary 4.26**

*If  $\gamma$  is free but not a free segment, then  $\underline{\gamma}$  either admits a  $\tau$ -decomposition or some compact maximal interval.*

Then, for  $\tau \in \text{int}[D]$  as well as  $D_- := D \cap (-\infty, \tau]$  and  $D_+ := [\tau, \infty)$ ,

- Let us write  $g \cdot \gamma|_{D_-} \rightarrow \gamma|_{D_+}$  iff  $g \cdot \gamma|_{B_-} = \gamma|_{B_+} \circ \mu$  holds for some (necessarily unique) analytic diffeomorphism  $\mu: B_- \rightarrow B_+$  with  $\mu(\tau) = \tau \in B_\pm$ , as well as  $B_- = D_-$  or  $B_+ = D_+$ .
- Let us say that  $[g]$  is a  $\tau$ -decomposition of  $\gamma$  iff  $D_\pm$  is free, iff we have  $[g] \neq [e]$ , and iff  $g \cdot \gamma|_{D_-} \rightarrow \gamma|_{D_+}$  holds w.r.t.  $\mu$ .<sup>32</sup> We say that  $[g]$  is **faithful** iff  $g' \cdot \gamma|_{D_-} \sim_\circ \gamma$  w.r.t.  $\rho$ , implies that either

$$[g'] = [e] \quad \text{and} \quad \bar{\rho}|_{D_-} = \text{id}_{D_-} \quad \text{or} \quad [g'] = [g] \quad \text{and} \quad \bar{\rho}|_{\text{dom}[\mu]} = \mu \quad \text{holds.}$$

Obviously,  $D_-$  and  $D_+$  are free iff  $(i', \tau]$  and  $[\tau, i)$  are free, and from our discussions in Subsection 2.3, we easily conclude that we have

$$g \cdot \gamma|_{D_-} \rightarrow \gamma|_{D_+} \quad \Longleftrightarrow \quad g \cdot \gamma|_{(i', \tau]} \rightsquigarrow \gamma|_{[\tau, i)}.$$

In fact, if the left hand side holds w.r.t.  $\mu$ , the right hand side holds w.r.t.

- $\underline{\mu} = \mu|_{(i', \tau]}$  if  $B_- = D_-$ ,
- $\underline{\mu} = \mu|_{(b', \tau]}$  if  $B_- \subset D_-$  is of the form  $(b', \tau]$  or  $[b', \tau]$ .

<sup>31</sup>Indeed, only in  $\tau$ -decompositions, two half-open intervals are involved, and only in  $A$ -decompositions, combinations of two compact as well as combinations of a compact with a half-open interval are involved.

<sup>32</sup>Then,  $D_\pm$  are the only maximal intervals, just by the same arguments as in Lemma 4.6.1.

Conversely, if the right hand side holds w.r.t.  $\underline{\mu}$ , then the left hand side holds w.r.t.

- $\mu = \underline{\mu}|_{D_-}$  if  $\text{im}[\underline{\mu}] \subset [\tau, i)$ ,
- $\mu = \underline{\mu}$  if  $\text{im}[\underline{\mu}] = [\tau, i) = D_+$ ,
- $\mu = \underline{\mu}|_C$  if  $\text{im}[\underline{\mu}] = [\tau, i)$  and  $D_+ = [\tau, i]$ , for  $C$  the closure of  $\text{dom}[\underline{\mu}]$  in  $D_-$ ,

whereby  $\underline{\mu}$  denotes the maximal analytic immersive extension of  $\underline{\mu}$ . Consequently,  $[g]$  is a  $\tau$ -decomposition of  $\gamma$  iff it is a  $\tau$ -decomposition of  $\underline{\gamma}$ , hence

- uniquely determined iff it exists by Lemma 4.6.3,
- faithful w.r.t.  $\gamma$  as it is so w.r.t.  $\underline{\gamma}$ , because  $g' \cdot \gamma|_{D_-} \sim_{\circ} \gamma$  w.r.t.  $\rho$  implies  $g' \cdot \underline{\gamma}|_{(i', \tau]} \sim_{\circ} \underline{\gamma}$  w.r.t.  $\rho$ .

Thus, we have shown that

#### Corollary 4.27

*If  $\gamma$  is free but not a free segment, it either admits a unique  $\tau$ -decomposition or some compact maximal interval contained in the interior of its domain.*

We finally have to discuss the situation where  $\underline{\gamma}$  admits a compact maximal interval  $A = [a', a]$ . In this case,  $A$  is maximal w.r.t.  $\overline{\gamma}$  as well, just by Lemma 4.9.2 applied to the identity on  $A$ . Then, by uniqueness, the  $A$ -decomposition  $\overline{\alpha} := (\{a_n\}_{n \in \mathbf{n}}, \{[g_n]\}_{n \in \mathbf{n}})$  of  $\overline{\gamma}$  restricts to the  $A$ -decomposition  $\underline{\alpha}$  of  $\underline{\gamma}$ , just by removing such indices  $n$  from  $\mathbf{n}$ , for which  $a_n \notin I$  holds.<sup>33</sup> Now,

- For analytic immersions  $\delta: A \rightarrow M$  and  $\delta': B \rightarrow M$ , with  $B$  of the form  $[b', b]$ ,  $[b', b)$  or  $(b', b]$ , let us write  $\delta \rightarrow \delta'$  iff  $\delta|_{A'} = \delta' \circ \rho$  holds for some analytic diffeomorphism  $\rho: A \supseteq A' \rightarrow B$  with  $A' \cap \{a', a\} \neq \emptyset$ . This diffeomorphism is necessarily unique iff it exists. For  $B$  half-open, this is clear from the uniqueness discussion in the beginning of Subsection 2.3 and Corollary 2.9; and if  $B$  is compact, additionally from Corollary 4.17.
- If  $A \subseteq \text{int}[D]$  is compact and free, an  $A$ -decomposition of  $\gamma$  is a pair  $(\{a_n\}_{n \in \mathbf{n}}, \{[g_n]\}_{n \in \mathbf{n}})$  with  $\{g_n\}_{n \in \mathbf{n}} \subseteq G$ , and  $\{a_n\}_{n \in \mathbf{n}}$  a decomposition of  $D$ , such that  $A_0 = A$  and  $[g_{\pm 1}] \neq [e]$  holds. In addition to that, we require that  $g_n \cdot \gamma|_A \rightsquigarrow \gamma|_{A_n}$  holds for all  $n \in \mathbf{n} \setminus \{\mathbf{n}_-, \mathbf{n}_+\}$ , as well as

$$g_{\mathbf{n}_-} \cdot \gamma|_A \rightarrow \gamma|_{A_{\mathbf{n}_-}} \quad \text{if } \mathbf{n}_- \neq -\infty \quad \text{and} \quad g_{\mathbf{n}_+} \cdot \gamma|_A \rightarrow \gamma|_{A_{\mathbf{n}_+}} \quad \text{if } \mathbf{n}_+ \neq \infty$$

holds. The respective analytic diffeomorphisms will be denoted by  $\mu_n$  for each  $n \in \mathbf{n}$ , whereby we define  $\mu_0 := \text{id}_A$  and  $g_0 := e$ . We will say that  $(\{a_n\}_{n \in \mathbf{n}}, \{[g_n]\}_{n \in \mathbf{n}})$  is **faithful** iff

$$g \cdot \gamma|_A \sim_{\circ} \gamma \quad \text{w.r.t. } \rho: J \rightarrow J' \quad \implies \quad [g] = [g_n] \quad \text{and} \quad \overline{\rho}|_{\text{dom}[\mu_n]} = \mu_n \quad \text{for } n \in \mathbf{n} \sqcup \{0\} \text{ unique.}$$

Then, for  $\overline{\alpha}$  and  $\underline{\alpha}$  the  $A$ -decompositions of  $\overline{\gamma}$  and  $\underline{\gamma}$ , respectively,  $\overline{\alpha}$  obviously restricts to an  $A$ -decomposition  $\alpha$  of  $\gamma$  as well, just in the same way we have described above for  $\underline{\gamma}$ . In particular, the decomposition  $\underline{\alpha}$  of  $\underline{\gamma}$  just arises from  $\alpha$  by restricting the diffeomorphisms  $\mu_n$  in the obvious way. Thus,

- $\alpha$  is unique as  $\underline{\alpha}$  is unique,
- $\alpha$  is faithful as  $\overline{\alpha}$  is faithful.

Finally, let us define  $\gamma$  to be **positive/negative** iff  $\underline{\gamma}$  is positive/negative. Then, since  $\overline{\gamma}$  is positive/negative if  $\underline{\gamma}$  is positive/negative,

i) We obtain from Proposition 4.19 applied to  $\overline{\gamma}$ :

If  $\gamma$  is positive, this proposition holds for  $\gamma$  in the sense that the right hand side of (39) holds for  $n \in \mathbf{n} \setminus \{\mathbf{n}_-, \mathbf{n}_+\}$ , and reads

$$g_{\mathbf{n}_{+/-}} \cdot \gamma|_A \rightarrow \gamma|_{A_{\mathbf{n}_{+/-}}} \quad \text{for } \dot{\mu}_{\mathbf{n}_{+/-}} > 0 \quad \text{if } \mathbf{n}_{+/-} \neq +/ -\infty \quad \text{holds.}$$

Moreover, the last statement in Proposition 4.19 holds for all  $t \in \text{int}[D]$ .

<sup>33</sup>Of course, the diffeomorphisms  $\mu_n$  then have to be restricted in the obvious way.

ii) We obtain from Proposition 4.22 applied to  $\overline{\gamma}$ :

If  $\gamma$  is negative, this proposition holds for  $\gamma$  in the sense that the left hand side of (43) is true for  $n \in \mathbf{n} \setminus \{\mathbf{n}_-, \mathbf{n}_+\}$ , and reads

$$g_{\mathbf{n}_{+/-}} \cdot \gamma|_A \rightarrow \gamma|_{A_{\mathbf{n}_{+/-}}} \quad \text{if} \quad \mathbf{n}_{+/-} \neq +/\infty,$$

whereby  $\mu_{\mathbf{n}_{+/-}}$  is positive/negative iff  $\mathbf{n}_{+/-}$  is even/odd.

Thus, Theorem 4.23 holds also in the form:

**Theorem 4.28**

*If  $\gamma: D \rightarrow M$  is free but not a free segment, it either admits a unique  $\tau$ -decomposition or a compact maximal interval contained in  $\text{int}[D]$ . In the second case,  $\gamma$  is either positive or negative, with what the statements in i) or ii) hold, respectively.*

In addition to that, we can apply Corollary 4.24 to  $\overline{\gamma}$ , in order to conclude that

**Corollary 4.29**

*Let  $\varphi$  be free and sated, and  $\gamma: D \rightarrow M$  a free curve. Then, for each  $t \in D$ , we find an open interval  $J \subseteq \mathbb{R}$  containing  $t$ , such that  $g \cdot \gamma(J \cap D) \cap \gamma(J \cap D)$  is finite for all  $g \in G \setminus \{e\}$ .*

## 5 Extension: Analytic 1-Manifolds

Besides the issues, we have discussed in Remark 3.8, it is an interesting observation that, given a connected analytic 1-manifold  $S$  with boundary together with an injective analytic immersion  $\iota: S \rightarrow M$ , each chart  $(U, \psi)$  of  $S$  defines the analytic immersive curve<sup>34</sup>

$$\gamma_\psi: \psi(U) \rightarrow M, \quad t \mapsto \iota \circ \psi^{-1}(t).$$

Thus, one might ask the question, whether the results of the previous sections carry over to  $S$ . In fact, defining  $(S, \iota)$  to be **Lie** iff  $\gamma_\psi$  is Lie for some chart  $(U, \psi)$ , we easily obtain that

**Proposition 5.1**

*If  $\varphi$  is sated and  $(S, \iota)$  is Lie, each  $\gamma_\psi$  is Lie with respect to the same  $x \in \text{im}[\iota]$  and  $\vec{g} \in \mathfrak{g} \setminus \mathfrak{g}_x$ . Then,  $(S, \iota)$  is either analytically diffeomorphic to  $U(1)$  or to some interval  $D \subseteq \mathbb{R}$  via*

$$U(1) \rightarrow S, \quad e^{i\phi} \mapsto \iota^{-1}(\exp(\phi \cdot \vec{g}) \cdot x) \quad \text{or} \quad D \rightarrow S, \quad t \mapsto \iota^{-1}(\exp(t \cdot \vec{g}) \cdot x), \quad (50)$$

*respectively, for  $\vec{g}$  rescaled in such a way that  $\pi_{\vec{g}} = 2\pi$  holds in the first case. Here,  $\vec{g}$  is unique up to addition of an element in  $\mathfrak{g}_S$ , provided that we fix  $D$  in the second case.*

Of course, here  $D$  and  $U(1)$  are meant to carry their standard analytic structures; and  $\mathfrak{g}_S$  denotes the Lie algebra of the stabilizer  $G_S := \bigcap_{z \in S} \iota(z)$  of  $S$ , which obviously coincides with the stabilizer of the curve  $\gamma_{\vec{g}}$ .

PROOF: The last statement is just clear from our discussions in the end of Subsection 2.5.2.

Now, by assumption, we find some chart  $(U_0, \psi_0)$  with  $\gamma_{\psi_0} = \gamma_{\vec{g}}^x \circ \rho_0$  for some  $x \in \text{im}[\iota]$ ,  $\vec{g} \in \mathfrak{g} \setminus \mathfrak{g}_x$ , and some analytic diffeomorphism  $\rho_0: \psi(U_0) \rightarrow D_0 \subseteq \mathbb{R}$ . Moreover, if  $\pi_{\vec{g}} < \infty$  holds, rescaling  $\vec{g}$  and  $\rho$  if necessary, we can assume that  $\pi_{\vec{g}} = 2\pi$  holds.

Then, the claim follows, if we show the existence of an interval  $D \subseteq \mathbb{R}$  with  $\gamma_{\vec{g}}^x(D) = \iota(S)$ , such that for each  $\gamma_\psi$ , we find some analytic diffeomorphism  $\rho$ , with  $\text{im}[\rho] \subseteq D$  and  $\gamma_\psi = \gamma_{\vec{g}}^x \circ \rho$ . In fact, then each  $\gamma_\psi$  is Lie w.r.t.  $x$  and  $\vec{g}$ , and

- If  $\gamma_{\vec{g}}^x|_D$  is injective, then  $\Omega: D \rightarrow S$  defined by the right hand side of (50) is bijective; as well as an analytic diffeomorphism, because for each chart  $(U, \psi)$  of  $S$ , we have

$$\Omega^{-1} \circ \psi^{-1} = (\gamma_{\vec{g}}^x|_D)^{-1} \circ \iota \circ \psi^{-1} = (\gamma_{\vec{g}}^x|_D)^{-1} \circ \gamma_\psi = \rho.$$

<sup>34</sup>If  $(U, \psi)$  is a chart of  $S$ , in the following, we will assume that  $U$  is open and connected, and that  $\iota(U)$  is contained in the domain some chart of  $M$ .

- If  $\gamma_{\bar{g}}^x|_D$  is not injective, we have  $\pi_{\bar{g}} = 2\pi$ , so that  $\Omega$  given by the left hand side of (50) is well defined and bijective. In addition to that,  $\Omega$  is an analytic diffeomorphism, because

$$\Omega^{-1} \circ \psi^{-1} = \Omega^{-1} \circ \iota^{-1} \circ \gamma_{\psi} = (\iota \circ \Omega)^{-1} \circ \gamma_{\bar{g}}^x \circ \rho = e^{i\rho}.$$

Now, let us fix some  $y \in U_0$ , and choose  $t \in D_0$  with  $\iota(y) = \gamma_{\bar{g}}^x(t)$ . Then, for existence of  $D$ , it suffices to show that for each chart  $(U, \psi)$ , there exists some interval  $D_{\psi}$  with  $t \in D_{\psi}$  and  $\gamma_{\bar{g}}^x(D_{\psi}) \subseteq \iota(S)$ , such that  $\gamma_{\psi} = \gamma_{\bar{g}}^x \circ \rho$  holds for some analytic diffeomorphism  $\rho$  with  $\text{im}[\rho] \subseteq D_{\psi}$ . In fact, then we can just define  $D$  to be the union of all  $D_{\psi}$  for  $(U, \psi)$  a chart of  $S$ . Now,

- By connectedness of  $S$ , for each  $z \in S$ , we find finitely many charts  $(U_1, \psi_1), \dots, (U_n, \psi_n)$ , such that  $z \in U_n$  and  $U_{i+1} \cap U_i \neq \emptyset$  holds for  $i = 0, \dots, n-1$ .

In fact, the set  $O$  of all  $z \in S$ , for which this statement holds is non-empty and open, just because the domain of each chart is open by convention. Then,  $O$  is also closed, because for  $z' \in S \setminus O$ , the domain of each chart around  $z'$  must completely be contained in  $S \setminus O$  as well.

- Thus, for each chart  $(U, \psi)$  of  $S$ , we find finitely many charts  $(U_1, \psi_1), \dots, (U_{n-1}, \psi_{n-1})$ , such that  $U_{i+1} \cap U_i \neq \emptyset$  holds for  $i = 0, \dots, n-1$  for  $(U_n, \psi_n) := (U, \psi)$ .

Then the claim follows if we show that in the situation of the second point, for each  $0 \leq i \leq n$ , we find an interval  $D_i$  with  $t \in D_i$  and  $\gamma_{\bar{g}}^x(D_i) \subseteq \iota(S)$ , such that  $\gamma_{\psi_i} = \gamma_{\bar{g}}^x \circ \rho_i$  holds for some analytic diffeomorphism  $\rho_i$  with  $\text{im}[\rho_i] \subseteq D_i$ . This is clear for  $i = 0$ , so that we can assume that it holds for some  $0 \leq i < n$ . Then,  $U_{i+1} \cap U_i \neq \emptyset$  implies

$$\gamma_{\psi_{i+1}} \sim \gamma_{\psi_i} \implies \gamma_{\psi_{i+1}} \sim \gamma_{\bar{g}}^x|_{\text{im}[\rho_i]} \implies \gamma_{\psi_{i+1}} \sim \gamma_{\bar{g}}^x|_{D_i} \xrightarrow{\text{Lemma 2.22}} \gamma_{\psi_{i+1}} = \gamma_{\bar{g}}^x \circ \rho_{i+1}$$

for some analytic diffeomorphism  $\rho_{i+1}$  with  $\text{im}[\rho_{i+1}] \cap D_i \neq \emptyset$ , by Lemma 2.22. Thus, the statement holds for  $i+1$  for  $\rho_{i+1}$  and  $D_{i+1} := \text{im}[\rho_{i+1}] \cup D_i$ , so that the claim follows inductively. ■

Next, let us say that  $(S, \iota)$  is free iff  $\gamma_{\psi}$  is free for some chart  $(U, \psi)$ . Then, Proposition 5.1 and Theorem 3.6 show that

### Corollary 5.2

If  $\varphi$  is regular,  $(S, \iota)$  is either free or Lie, with what each  $\gamma_{\psi}$  is free or Lie, respectively.

Next, observe that the stabilizer  $G_S := \bigcap_{z \in S} G_{\iota(z)}$  of  $S$  coincides with the stabilizer of each  $\gamma_{\psi}$ , because

- For  $(U_0, \psi_0)$  fixed, and  $(U, \psi)$  another chart of  $S$ , we find charts  $(U_1, \psi_1), \dots, (U_{n-1}, \psi_{n-1})$  with  $U_{i+1} \cap U_i \neq \emptyset$  for  $i = 0, \dots, n-1$  for  $(U_n, \psi_n) := (U, \psi)$ , cf. proof of Proposition 5.1.
- Then, for each such  $0 \leq i \leq n-1$ , we have

$$U_{i+1} \cap U_i \neq \emptyset \implies \gamma_{\psi_{i+1}} \sim \gamma_{\psi_i} \xrightarrow{\text{Lemma 2.18}} G_{\gamma_{\psi_{i+1}}} = G_{\gamma_{\psi_i}},$$

hence  $G_{\gamma_{\psi}} = G_{\gamma_{\psi_0}}$ , from which the claim is clear.

Then, if each chart of  $S$  is free, we denote by  $M$  the set of all  $z \in S$ , for which we find a chart  $(U, \psi)$  with

$$g \cdot \bar{\gamma}_{\psi}|_{(-\infty, \psi(z)] \cap J} \rightsquigarrow \psi(z), \psi(z) \bar{\gamma}_{\psi}|_{[\psi(z), \infty) \cap J} \quad (51)$$

for some  $g \in G \setminus G_S$ , and some open interval  $J \subseteq \text{dom}[\bar{\gamma}_{\psi}]$  with  $\psi(z) \in J$ . Then,

### Lemma 5.3

If  $\psi(z)$  is contained in the interior of some free interval in  $\text{dom}[\bar{\gamma}_{\psi}]$  for some chart  $(U, \psi)$ , we have  $z \in S \setminus M$ . In particular,  $U \cap M$  is at most countable for each chart  $(U, \psi)$  of  $S$  by Theorem 4.23.

PROOF: Let  $J \subseteq \text{dom}[\bar{\gamma}_{\psi}]$  be free and open with  $\psi(z) \in J$ . Then, if the statement is wrong, we find a chart  $(U', \psi')$  around  $z$ , some  $g' \in G \setminus G_S$ , and  $J' \subseteq \text{dom}[\bar{\gamma}_{\psi'}]$ , such that (51) holds for  $\psi'$ ,  $J'$ ,  $g'$  and  $z$ . Since we have  $z \in U \cap U'$ , Lemma 2.4 shows that  $\bar{\gamma}_{\psi}|_I = \bar{\gamma}_{\psi'} \circ \rho$  holds for some analytic diffeomorphism  $\rho: I \rightarrow I'$  with  $\psi(z) \in I$ ,  $\psi'(z) \in I'$ , and  $\rho(\psi(z)) = \psi'(z)$ . This implies  $g' \cdot \bar{\gamma}_{\psi}|_J \sim \bar{\gamma}_{\psi}|_J$ , which contradicts that  $J$  is free, and that  $g' \notin G_S = G_{\psi}$  holds. ■

From this, we easily obtain

#### Corollary 5.4

If  $\varphi$  is regular and  $(S, \iota)$  is free, then  $M$  is at most countable, and even empty if  $\varphi$  is in addition free. Moreover, each  $z \in S \setminus M$  admits a neighbourhood  $V \subseteq S$ , such that  $g \cdot \iota(V) \cap \iota(V)$  is finite for all  $g \in G \setminus G_S$ .

PROOF: If  $\varphi$  is free, we must have  $M = \emptyset$ , just because  $G_z$  is trivial for each  $z \in S$ . In the general case, we can cover  $S$  by countably many charts, and conclude from Lemma 5.3 that  $M$  must be countable.

Now, for  $z \in S \setminus M$  and  $(U, \psi)$  some chart with  $z \in U$ , by Theorem 4.23,  $\psi(z)$  must be contained in some free open interval  $J' \subseteq \text{dom}[\overline{\gamma}_\psi]$ . Then, shrinking  $J'$  if necessary, we can assume that  $\overline{\gamma}_\psi|_{J'}$  is an embedding. We choose some compact neighbourhood  $K \subseteq J'$  of  $\psi(z)$ , and define  $J := \text{int}[K]$ , as well as  $V := \psi^{-1}(J \cap \text{im}[\psi])$ . Then, if  $g \cdot \iota(V) \cap \iota(V)$  is infinite, the same is true for  $g \cdot \overline{\gamma}_\psi(J) \cap \overline{\gamma}_\psi(J)$ , so that Lemma 2.3 shows that

$$g \cdot \overline{\gamma}_\psi|_{J'} \sim_\circ \overline{\gamma}_\psi|_{J'} \implies g \in G_{\overline{\gamma}_\psi} = G_{\gamma_\psi} = G_S. \quad \blacksquare$$

Now,

#### Remark 5.5

The last statement in the above corollary also holds if  $z \in M$  is a boundary point of  $S$ . In fact,

- ▷ By assumption, (51) holds for some boundary chart  $(U, \psi)$  around at  $z$  with  $\psi(U) \subseteq (-\infty, 0]$  and  $\psi(z) = 0$ . Then,  $[g]$  is a 0-decomposition of  $\overline{\gamma}_\psi|_J$ , for  $J$  some suitably small open interval containing  $\psi(z)$ . This is clear from Theorem 4.23, because  $\psi(z)$  cannot be contained in the interior of any free interval in  $\text{dom}[\overline{\gamma}_\psi]$  by Lemma 5.3.
- ▷ Then, shrinking  $J = (j', j)$  around 0 if necessary, we can assume that  $(j', 0] \subseteq \psi(U)$  and  $g \cdot \overline{\gamma}_\psi((j', 0]) = \overline{\gamma}_\psi([0, j))$  holds, and that  $\overline{\gamma}_\psi|_J$  is an embedding.
- ▷ The claim now holds for  $V := \psi^{-1}((k, 0])$  for  $j' < k < 0$ , because  $g' \cdot \iota(V) \cap \iota(V)$  infinite implies  $g' \cdot \overline{\gamma}_\psi|_{(j', 0]} \sim_\circ \overline{\gamma}_\psi|_J$ , hence  $[g'] = [e]$  or  $[g'] = [g]$  and  $g' \cdot \overline{\gamma}_\psi((j', 0]) = \overline{\gamma}_\psi([0, j))$  by faithfulness. In the second case, however,  $g' \cdot \iota(V) \cap \iota(V)$  cannot be infinite by injectivity of  $\overline{\gamma}_\psi|_J$ , so that  $[g'] = [e]$ , hence  $g' \in G_{\overline{\gamma}_\psi} = G_{\gamma_\psi} = G_S$  must hold.  $\ddagger$

Finally, in order to obtain global decomposition results also for connected 1-manifolds, it seems to be reasonable to make the definitions more similar to that ones, we have used for analytic immersive curves. For instance, we can define  $(S, \iota)$  to be **free** iff it admits a **free segment**, i.e., a connected subset  $\Sigma \subseteq S$  with non-empty interior, such that

$$g \cdot \iota|_\Sigma \sim_\circ \iota|_\Sigma \quad \text{for } g \in G \implies g \in G_S.$$

Here, we write  $g \cdot \iota|_\Sigma \sim_\circ \iota|_\Sigma$  for segments  $\Sigma, \Sigma' \subseteq S$  iff  $g \cdot \iota(\mathcal{O}) = \iota(\mathcal{O}')$  holds for open segments  $\mathcal{O} \subseteq \Sigma$  and  $\mathcal{O}' \subseteq \Sigma'$  that are contained in the interior of  $S$ , and on which  $\iota$  is an embedding. Then,  $(S, \iota)$  is free in the sense of our new definition iff it is free in the sense of our former one. We define a free segment  $\Sigma$  to be maximal iff  $\Sigma \subseteq \Sigma'$  for a free segment  $\Sigma' \subseteq S$ , implies  $\Sigma = \Sigma'$ . Since  $S$  is either homeomorphic to an interval or to  $U(1)$ , it is easy to see that  $\Sigma$  is closed, and that the following analogue to Lemma 4.2 holds.

#### Lemma 5.6

If  $\Sigma \subseteq S$  is a free segment, we find  $\Sigma' \subseteq S$  maximal with  $\Sigma \subseteq \Sigma'$ .

One strategy here now can be, first to consider such  $(S, \iota)$  without boundary, and then to carry over the results to the boundary case, just by considering the interior of  $S$ . More precisely, one can define the classes  $[g] := g \cdot G_S$ , and then adapt Proposition 4.11 to the 1-manifold case. Then, one has to go through the arguments of Section 4, keeping in mind that  $S$  can be compact now. Indeed, if  $\varphi$  is sated, and  $(S, \iota)$  is free without boundary, it is to be expected that:

If  $S$  is non-compact, and not a free segment by itself, then<sup>35</sup>

<sup>35</sup>Observe that analytic embedded curves are both analytic immersive curves and connected analytic 1-submanifolds; and then the first two situations described below, just encode the  $\tau$ - and the  $A$ -decomposition case, we have discussed in this paper.

- If  $S$  admits no compact maximal segment, it admits only two maximal segments  $\Sigma, \Sigma'$ , and we have  $S = \Sigma \cup \Sigma'$  as well as  $\Sigma \cap \Sigma' = \{z\}$  for  $z \in S$  unique. In addition to that, either  $g \cdot \iota(\Sigma) \subseteq \iota(\Sigma')$  or  $\iota(\Sigma') \subset g \cdot \iota(\Sigma)$  holds for some element  $g \in G_z$ . Here, the class of  $g$  is uniquely determined by the property that  $[g] \neq [e]$  as well as  $g \cdot \iota|_{\Sigma} \sim_{\circ} \iota$  holds.
- If  $S$  admits some compact maximal  $\Sigma_0$ , there exists a  $\Sigma_0$ -decomposition  $\mathcal{S}$  of  $S$ ; i.e., a family  $\{(\Sigma_n, [g_n])\}_{n \in \mathbf{n}}$  consisting of free segments  $\Sigma_n$  on which  $\iota$  is an embedding, as well as classes  $[g_n]$ , such that
  - $\Sigma_m \cap \Sigma_n \neq \emptyset$  is singleton for  $|m - n| = 1$ , and empty otherwise,
  - $\Sigma_n$  is compact for  $\mathbf{n}_- < n < \mathbf{n}_+$ ,
  - $g_n \cdot \iota(\Sigma_0) \supseteq \iota(\Sigma_n)$  holds for all  $n \in \mathbf{n}$ , whereby a proper inclusion only holds for  $n = \mathbf{n}_-$  if  $\mathbf{n}_- \neq -\infty$  and  $n = \mathbf{n}_+$  if  $\mathbf{n}_+ \neq +\infty$ .

The only other  $\Sigma_0$ -decomposition  $\overline{\mathcal{S}}$  of  $S$ , is given by

$$(\{\overline{\Sigma}_n\}_{n \in \overline{\mathbf{n}}}, \{\overline{g}_n\}_{n \in \overline{\mathbf{n}}}) \quad \text{with} \quad \overline{\Sigma}_n := \Sigma_{-n} \quad \text{and} \quad [\overline{g}_n] := [g_{-n}] \quad \text{for each} \quad n \in \mathbf{n},$$

whereby  $\overline{\mathbf{n}} := \{n \in \mathbb{Z}_{\neq 0} \mid \overline{\mathbf{n}}_- \leq n \leq \overline{\mathbf{n}}_+\}$  holds for  $\overline{\mathbf{n}}_{\pm} := -\mathbf{n}_{\mp}$ .

Moreover, for  $z_{\pm}$  the boundary points shared by  $\Sigma_{\pm 1}$  and  $\Sigma_0$ , we either have  $g_{\pm 1} \notin G_{z_{\pm}}$  or  $g_{\pm 1} \in G_{z_{\pm}}$ . In the first case, we will say that  $\Sigma_0$  is **positive**, and the second one that  $\Sigma_0$  is **negative**. Moreover, for  $\kappa: S \rightarrow I$  some fixed homeomorphism with  $I$  some open interval, we will say that  $\mathcal{S}$  is  $\kappa$ -oriented iff  $\kappa(z_-) < \kappa(z_+)$  holds. Then,

- If  $\Sigma_0$  is positive, each other compact maximal segment is positive, and each point in  $S$  is contained in the interior of such a positive segment. Moreover,  $[g_n] = [h^n]$  holds for all  $n \in \mathbf{n}$ , for some unique class  $[h]$ . This class is independent on the maximal segment  $\Sigma_0$ , provided that the respective  $\kappa$ -oriented decomposition of  $S$  is chosen.<sup>36</sup>
- If  $\Sigma_0$  is negative, the segments  $\Sigma_n$  for  $n \in \mathbf{n} \sqcup \{0\}$  are maximal, and the only maximal segments of  $S$ . Each compact  $\Sigma_n$  is negative, and  $[g_n] = [g_{\sigma(\text{sign}(n))} \cdots g_{\sigma(n)}]$  holds for each  $n \in \mathbf{n}$ , for  $\sigma$  defined by (42).  $\ddagger$

Now, if  $S$  is compact, it must admit some compact maximal segment  $\Sigma_0$ , just by existence and closedness of such segments. Then, if  $S$  is not a free segment by itself, it is to be expected that:

There exists a  $\Sigma_0$ -decomposition  $\mathcal{S}$  of  $S$ ; i.e., compact maximal segments  $\Sigma_1, \dots, \Sigma_n$  with  $S = \Sigma_0 \cup \dots \cup \Sigma_n$  for  $n \geq 1$ , as well as classes  $[g_1], \dots, [g_n]$ , such that  $g_k \cdot \iota(\Sigma_0) = \iota(\Sigma_k)$  holds for all  $k = 1, \dots, n$ , and

- If  $n = 1$ , then  $\Sigma_0 \cap \Sigma_1$  consists of two elements,
- If  $n > 1$ , then  $\Sigma_p \cap \Sigma_q$  is singleton for  $|p - q| \in \{1, n\}$ , and empty otherwise.

Here, for  $n = 1$ , only one  $\Sigma_0$ -decomposition exists, and for  $n \geq 2$ , the only other  $\Sigma_0$ -decomposition  $\overline{\mathcal{S}}$  of  $S$ , is given by

$$\overline{\Sigma}_k := \Sigma_{\zeta(k)} \quad \text{and} \quad [\overline{g}_k] := [g_{\zeta(k)}] \quad \forall 1 \leq k \leq n, \quad (52)$$

for  $\zeta \in S_n$  defined by  $\zeta(k) := n - (k - 1)$  for  $k = 1, \dots, n$ .

Now, let  $z_{\pm}$  denote the boundary points of  $\Sigma_0$ , such that  $\{z_+\} = \Sigma_0 \cap \Sigma_1$  holds for  $n \geq 2$ . Moreover, for  $\kappa: U(1) \rightarrow S$  a fixed homeomorphism, let us say that  $\mathcal{S}$  is  $\kappa$ -oriented iff  $\Sigma_0 = \kappa(e^{i[\alpha_-, \alpha_+]})$  as well as  $\kappa(\alpha_{\pm}) = z_{\pm}$  holds for some  $\alpha_- < \alpha_+$ .

Then,  $\Sigma_0$  is either **positive** or **negative** (same definition as above), and we have

- If  $\Sigma_0$  is positive, each other maximal segment  $\Sigma'_0$  is positive, and each point in  $S$  is contained in the interior of some positive segment. Moreover, for  $\mathcal{S}'$  a  $\Sigma'_0$ -decomposition of  $S$  with respective maximal segments  $\Sigma'_1, \dots, \Sigma'_{n'}$  and classes  $[g'_1], \dots, [g'_{n'}]$ , we have  $n' = n$  and  $[g'_k] = [h^k]$  for  $k = 1, \dots, n$ . Here, the class  $[h]$

<sup>36</sup>Then, for the only other  $\Sigma_0$ -decomposition  $\overline{\mathcal{S}}$  of  $S$ , the mentioned formula holds for  $\overline{h} := h^{-1}$  instead of  $h$ .



is unique, and the same for each positive  $\Sigma'_0$ , provided that the respective  $\kappa$ -oriented decomposition of  $S$  is chosen.

For instance, let  $S := U(1)$ , and define  $G$  to be the discrete subgroup of  $U(1)$  generated by  $h := e^{i2\pi/n}$ , just acting via multiplication from the left. Then,  $\Sigma = e^{iK}$  is positive for each  $K = [t, t + 2\pi/n]$ .

- If  $\Sigma_0$  is negative, the  $\Sigma_k$  are negative for  $k = 0, \dots, n$ , and the only maximal segments of  $S$ . Moreover,  $n$  is odd, and for  $n \geq 3$  and  $g_{-1} := g_n$ , we have  $[g_k] = [g_{\sigma(1)} \cdot \dots \cdot g_{\sigma(k)}]$  for  $k = 1, \dots, n$ .

For instance, let  $S := U(1) \subseteq \mathbb{R}^2$ , and  $G$  be the discrete group generated by the reflection at the  $x_2$ -axis. Then,  $\Sigma_0 = e^{iK_0}$  and  $\Sigma_1 = e^{iK_1}$  are negative for  $K_0 = [-\pi/2, \pi/2]$  and  $K_1 = [\pi/2, 3\pi/4]$ . Similarly, if  $G$  is the discrete group generated by the reflection at the  $x_1$ - and the  $x_2$ -axis, then  $\Sigma_i = e^{iK_i}$  is negative for  $K_i = [i \cdot \pi/4, (i+1) \cdot \pi/4]$  for  $i = 0, \dots, 3$ , and the above formula for the classes  $[g_i]$  is easily verified.  $\ddagger$

## Acknowledgements

The author thanks Chris Beetle, Rory Conboye, Jonathan Engle and Bernhard Krötz for their remarks on drafts of the present article. This work was supported in part by the Alexander von Humboldt foundation of Germany, and NSF Grants PHY-1205968 and PHY-1505490.

## References

- [1] A. Ashtekar, J. Lewandowski: Background Independent Quantum Gravity: A Status Report. *Class. Quant. Grav.* **21** (2004) R53-R152. e-print: 0404018v2 (gr-qc).
- [2] T. Thiemann: *Introduction to Modern Canonical Quantum General Relativity*. Cambridge University Press, 2008.
- [3] Ch. Fleischhack: Symmetries of Analytic Paths. e-print: arxiv:1503.06341 (math-ph).
- [4] M. Hanusch: Invariant Connections and Symmetry Reduction in Loop Quantum Gravity (Dissertation). University of Paderborn, December 2014. <http://nbn-resolving.de/urn:nbn:de:hbz:466:2-15277> e-print: arxiv:1601.05531 (math-ph).
- [5] M. Hanusch: Decompositions of Analytic 1-Manifolds.